Logical Implication

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We understand at least two rows of implication's truth table.

A	B	$A \to B$
F	F	?
F	T	?
T	F	F
T	T	T

Basically, "if the antecedent is true, the consequent must also be true". But what do we say about the entire implication if the antecedent is false?
The "shut up and follow the definition" approach tells us to consider the implication true.

A	B	$A \to B$
F	F	$\boxed{\texttt{T}}$
F	T	$\boxed{\texttt{T}}$
T	F	F
T	T	T

But why?

Intuitively

The textbook gives the example

> "If I pass my economics test, then I'll go to the movie on Friday."

If you pass the test and go to the movie, the implication is true. If you pass the test and do *not* go to the movie, the impication is false—passing the test was not a precondition to going to the movie. But if you don't pass the test, whether you go to the movie or not doesn't render the statement incorrect, per se. Since you haven't fulfilled the precondition (passing the test), you can't say whether or not going to the movie followed from having done so. Thus, we'll "go ahead" and say that the implication is still true.

Similar examples:

- "If it is overcast out, then the sun is invisible."
  - If it is overcast, it *must* be the case that the sun is invisible.
  - If it is not overcast, the sun may still be invisible (e.g., at night), but the implication still isn't false.
- "If I have \$5, then I will go to Subway."
  - Assuming you have a serious Subway habit, \$5 will definitely cause you to go to Subway.
  - However, you may go to Subway regardless of whether you have \$5 (e.g., your friend buys it for you, or you just don't go at all).

Formally

Let's look at what happens if we instead decide that a false antecedent means a false implication. I.e., let's use the following truth table. (We use the symbol $\xrightarrow{\star}$ to avoid confusion with actual implication ($\rightarrow$).)

A	B	$A \xrightarrow{\star} B$
F	F	$\boxed{\texttt{F}}$
F	T	$\boxed{\texttt{F}}$
T	F	F
T	T	T

But let's see what happens when we try to use $\xrightarrow{\star}$.
Generally speaking, we want equivalence ($A \leftrightarrow B$) to mean the same thing as

$$(A \rightarrow B) \wedge (B \rightarrow A)$$

because it is convenient to define it as such. However, with the altered definition of equivalence ($A \overset{\star}{\leftrightarrow} B$), we'll instead have

$$(A \xrightarrow{\star} B) \wedge (B \xrightarrow{\star} A)$$

This is a problem:

A	B	$A \xrightarrow{\star} B$	$B \xrightarrow{\star} A$	$(A \xrightarrow{\star} B) \wedge (B \xrightarrow{\star} A)$
F	F	F	F	F
F	T	F	F	F
T	F	F	F	F
T	T	T	T	T

As seen via the related definition of equivalence, the original truth table for $\rightarrow$ captures the intended meaning.
Further, notice that the truth table for $\xrightarrow{\star}$ is the same as for conjunction!