CS 130 Homework 2

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Practice Problems

You are not required to turn these in.

Unfortunately, the Gersting textbook takes a less axiomatic approach to logic, and *Book of Proof* doesn't even cover formal logic in depth. So, they won't be useful for practice. Instead, try proving the following theorems. Each proof may use any theorem with a lower number. (Additionally, these proofs might be easier to do after you finish the turn-in problems.)

$$
\begin{array}{rcll}
B & \vdash & (A \implies B) & (\textbf{Theorem 1}) \\
(A \implies (B \implies C)),\quad B & \vdash & (A \implies C) & (\textbf{Theorem 2}) \\
(A \implies (B \implies C)) & \vdash & (B \implies (A \implies C)) & (\textbf{Theorem 3}) \\
(P \implies R) & \vdash & (P \implies (Q \implies R)) & (\textbf{Theorem 4}) \\
 & \vdash & ((\neg B \implies \neg A) \implies (A \implies B)) & (\textbf{Theorem 5}) \\
 & \vdash & (\neg\neg B \implies B) & (\textbf{Theorem 6}) \\
 & \vdash & (B \implies \neg\neg B) & (\textbf{Theorem 7})
\end{array}
$$

Turn-in Problems

$\boxed{1}$ We've seen that the Deduction Theorem lets us (effectively) turn implications into inference rules. Let's do this to the three axioms of propositional logic.

(a) Apply the Deduction Theorem to Axiom 1 to get a new inference rule. Call this new rule **Axiom 1i**.

(b) Apply the Deduction Theorem to Axiom 2 to get a new inference rule. Call this new rule **Axiom 2i**.

(c) Apply the Deduction Theorem to Axiom 3 to get a new inference rule. Call this new rule **Axiom 3i**.

(d) Do we need to provide proofs for these new inference rules? Why or why not?

In principle, any tautology can be proven using only instances of Axioms 1–3 and Modus Ponens. That doesn't mean it's always convenient, though. We'll investigate this by proving the following inference rules in different ways:

$$(A \implies B),\quad (B \implies C) \quad\vdash\quad (A \implies C) \qquad \textbf{(Syllogism)}$$
$$\neg\neg A \quad\vdash\quad A \qquad \textbf{(Double Negation)}$$
$$(A \implies B),\quad (A \implies (B \implies C)) \quad\vdash\quad (A \implies C) \qquad \textbf{(Modus Ponens Deduction)}$$

$\boxed{2}$ (a) Give a fully-justified proof of Modus Ponens Deduction using only instances of Axioms 1–3 and Modus Ponens.

(b) Prove Modus Ponens Deduction is valid by using a truth table.
(*Hint:* apply the Deduction Theorem until Modus Ponens Deduction has no premises.)

$\boxed{3}$ (a) Justify each step in the following proof of Syllogism, assuming we can only use instances of Axioms 1–3 and Modus Ponens:

1. $(A \implies B)$
2. $(B \implies C)$
3. $((B \implies C) \implies (A \implies (B \implies C)))$
4. $(A \implies (B \implies C))$
5. $((A \implies (B \implies C)) \implies ((A \implies B) \implies (A \implies C)))$
6. $((A \implies B) \implies (A \implies C))$
7. $(A \implies C)$

(b) Prove Syllogism is valid by using a truth table.
(*Hint:* apply the Deduction Theorem until Syllogism has no premises.)

(c) Give a shorter proof of Syllogism than in part (a) by using the inference rules from $\boxed{1}$ and Modus Ponens Deduction.

$\boxed{4}$ (a) Despite being easy to state, Double Negation is difficult to prove symbolically. Even when we're "allowed" to use the inference rules we've learned so far, the proof sequence is long. Below is the sequence of justifications in such a proof. Write out the well-formed formula corresponding to each step.

1. Hypothesis
2. Axiom 1: $A := \neg\neg A$, $B := \neg A$
3. Axiom 3: $A := \neg A$, $B := A$
4. Syllogism: lines 2 and 3
5. Axiom 1: $A := \neg A$, $B := \neg\neg(\neg A \implies A)$
6. Axiom 3: $A := A$, $B := \neg(\neg A \implies A)$
7. Syllogism: lines 5 and 6
8. Axiom 2i: line 7
9. Axiom 3: $A := (\neg A \implies A)$, $B := A$
10. Syllogism: lines 8 and 9

11. Principle of Identity: $A := (\neg A \implies A)$
12. Modus Ponens Deduction: lines 11 and 10
13. Syllogism: lines 4 and 12
14. Modus Ponens: lines 1 and 13

(b) Prove Double Negation is valid by using a truth table.
(*Hint:* apply the Deduction Theorem until Double Negation has no premises.)

(c) We could make the proof in part (a) three lines shorter by introducing a new inference rule for a pattern that repeats itself three times in the original. Find that pattern, and state the new inference rule that would replace it.