CS 130 Homework 3

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DUE: January 27, 2012

Practice Problems

You are not required to turn these in.

Gersting, 6e: Section 1.3, Exercises 1–18, 24–26

Book of Proof: Section 2.7, Exercises 1–10

Turn-in Problems

Predicate logic is a lot like propositional logic, except we have the quantifiers $\forall$ and $\exists$. As such, predicate logic proofs will (ideally) be very similar to propositional logic proofs, except we have to contend with the quantifiers. Roughly, the overall strategy goes:

- Tear off quantifiers to get to the underlying propositions.
- Use propositional logic to mess with the formula.
- Wrap the result back up in the desired quantifiers.

We've got the middle step covered by our previous methods. This leaves us with the need to remove and add quantifiers to formulas.

$\boxed{1}$ To remove a $\forall x[\ldots]$ from around a formula, we have Axiom 4. But using it as an axiom and having to apply Modus Ponens is a bit clunky. Write an inference rule version of Axiom 4, and call it **Universal Instantiation**. Do you need to prove this inference rule is valid?

$\boxed{2}$ To wrap a $\forall x[\ldots]$ around a formula, we have the inference rule **Universal Generalization**. Use this together with Universal Instantiation from $\boxed{1}$ to prove the following:

$$\forall x[\forall y[A(x,y)]] \qquad \vdash \qquad \forall y[\forall x[A(x,y)]]$$

$\boxed{3}$ If a predicate is true for *everything*, it should be true for *something*. Since $\exists x[W]$ is a shortcut for $\neg\forall x[\neg W]$, we should be able to prove a theorem that lets us wrap a formula in $\exists x[\ldots]$:

$$A(t) \qquad \vdash \qquad \exists x[A(x)] \qquad\qquad (\textbf{Existential Generalization})$$

<u>Provided that:</u> $(\forall x[\neg A(x)] \implies \neg A(t))$ is a correct instance of Axiom 4.

Provide justifications for the following proof of Existential Generalization.

1. $A(t)$
2. $(\forall x[\neg A(x)] \implies \neg A(t))$
3. $\neg\neg A(t)$
4. $\neg\forall x[\neg A(x)]$

Why is the last line the desired conclusion?

$\boxed{4}$ Informally, if we have $\exists x[A(x)]$, we should be able to find *something* to plug in for x. If we give it a temporary name, we could plug it in and proceed with the rest of the proof. The best thing to use as a name is a constant symbol. But if we use one that has already appeared in the proof (or in some weird axioms we plan to use later), we will be implicitly making assumptions about that constant. So, we want to plug in a *new* constant when we remove $\exists$.

Formally, we state this as

$$\exists x[A(x)] \qquad \vdash \qquad A(\underline{c}) \qquad\qquad (\textbf{Existential Instantiation})$$

Provided that:

1. $\underline{c}$ is a new constant that hasn't appeared earlier in the proof or in any proper axioms we ever plan to use.
2. If some variable (say y) appears free in the formula $\exists x[A(x)]$, then Universal Generalization over y never occurs in the proof.

(a) Use Existential Instantiation and Existential Generalization to prove

$$\exists x[(A(x) \wedge B(x))] \qquad \vdash \qquad \exists x[A(x)]$$

(b) What is wrong with the following "proof" of

$$\exists x[A(x)], \quad \exists x[B(x)] \qquad \vdash \qquad \exists x[(A(x) \wedge B(x))]$$

which is *invalid* in general?

1. $\exists x[A(x)]$ — Hypothesis
2. $\exists x[B(x)]$ — Hypothesis
3. $A(\underline{c})$ — Existential Instantiation: Line 1
4. $B(\underline{c})$ — Existential Instantiation: Line 2
5. $(A(\underline{c}) \wedge B(\underline{c}))$ — Conjunction: Lines 3 and 4
6. $\exists x[(A(x) \wedge B(x))]$ — Existential Generalization: Line 5

(c) What is wrong with the following "proof" of

$$\forall y[A(y,y)] \qquad \vdash \qquad \exists x[\forall y[A(x,y)]]$$

which is *invalid* in general?

1. $\forall y[A(y,y)]$	Hypothesis
2. $A(y,y)$	Universal Instantiation: Line 1
3. $\exists x[A(x,y)]$	Existential Generalization: Line 2
4. $A(\underline{c},y)$	Existential Instantiation: Line 3
5. $\forall y[A(\underline{c},y)]$	Universal Generalization: Line 4
6. $\exists x[\forall y[A(x,y)]]$	Existential Generalization: Line 5