Combinatorics

CS 130

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Why Bother With Sets?

They facilitate formal discussions

- Naïve set theory
- Zermelo-Fraenkel set theory with the axiom of choice (ZFC)
- "Classes" circa George Boole's time

They're an important concept throughout computer science

- At a high level, collections of objects are everywhere
- The most basic collection is a set
- Mathematically, we can prove many things using sets
- They're an important concept in math for the same reasons

Multiple Choice Question

Suppose we have the set

$$\{a, b, c\}$$

How many different subsets of cardinality 2 are there?

(A) 2

(B) 3

(C) 8

(D) None of the above

Multiple Choice Question

Suppose we have the set

$$\{a, b, c, d\}$$

How many different subsets of cardinality 2 are there?

(A) 2

(B) 3

(C) 8

(D) None of the above

Multiple Choice Question

Let S be an arbitrary set with cardinality n.
What must be true about n?

(A) $n > 0$

(B) $n \geq 0$

(C) $n \in \mathbb{Z}$

(D) Can't conclude anything about n

Multiple Choice Question

Let S be an arbitrary set with cardinality n.
Suppose we have some $r \in \mathbb{N}$ such that $r \leq n$
Claim: The number of subsets of S with cardinality r is equal to

$$\frac{n!}{(n-r)!r!}$$

How should we prove this?

(A) A $\subseteq / \supseteq$ proof

(B) An $\implies / \impliedby$ proof

(C) Proof by Cases

(D) Proof by Induction

Multiple Choice Question

Let S be an arbitrary set with cardinality n.
Suppose we have some $r \in \mathbb{N}$ such that $r \leq n$
Claim: The number of subsets of S with cardinality r is equal to

$$\frac{n!}{(n-r)!r!}$$

Base Case.

. . . $\square$

What is the base case?

(A) When $n = 0$

(B) When $r = 0$

(C) When $n = 1$

(D) When $r = 1$

Multiple Choice Question

Let S be an arbitrary set with cardinality n.
Suppose we have some $r \in \mathbb{N}$ such that $r \leq n$
Claim: The number of subsets of S with cardinality r is equal to

$$\frac{n!}{(n-r)!r!}$$

Base Case ($n = 0$).

Assume $n = 0$. Then $r \ldots$ $\square$

What do we know about r at the base case?

(A) $r = 1$

(B) $r = 0$

(C) Either one of the above

(D) None of the above

Multiple Choice Question

Let S be an arbitrary set with cardinality n.
Suppose we have some $r \in \mathbb{N}$ such that $r \leq n$
Claim: The number of subsets of S with cardinality r is equal to

$$\frac{n!}{(n-r)!r!}$$

Base Case ($n = 0$).

Assume $n = 0$. Then $r = 0$, because $r \leq n$ and $r \in \mathbb{N}$.
$\frac{n!}{(n-r)!r!} = \cdots$ □

What is the result of plugging in $n = r = 0$ here?

(A) 0
(B) 1
(C) Undefined (division by zero)
(D) None of the above

Multiple Choice Question

Let S be an arbitrary set with cardinality n.
Suppose we have some $r \in \mathbb{N}$ such that $r \leq n$
Claim: The number of subsets of S with cardinality r is equal to

$$\frac{n!}{(n-r)!r!}$$

Base Case ($n = 0$).

Assume $n = 0$. Then $r = 0$, because $r \leq n$ and $r \in \mathbb{N}$.
$\frac{n!}{(n-r)!r!} = \frac{0!}{0!} = \frac{1}{1} = 1$
There are . . . ways of selecting an r-element subset from S. $\square$

When $n = r = 0$, how many r-element subsets of S are there?
(A) 0
(B) 1
(C) Undefined (division by zero)
(D) None of the above

Multiple Choice Question

Let S be an arbitrary set with cardinality n.
Suppose we have some $r \in \mathbb{N}$ such that $r \leq n$
Claim: The number of subsets of S with cardinality r is equal to

$$\frac{n!}{(n-r)!r!}$$

Base Case ($n = 0$).

Assume $n = 0$. Then $r = 0$, because $r \leq n$ and $r \in \mathbb{N}$.
$\frac{n!}{(n-r)!r!} = \frac{0!}{0!} = \frac{1}{1} = 1 \quad = \quad \#$ 0-element subsets of $S = \{\}$. $\square$

What do we assume for the inductive step?

(A) There are $\frac{k!}{(k-r)!r!}$ subsets with cardinality $r = 0$

(B) There are $\frac{k!}{(k-r)!r!}$ subsets with cardinality $r \leq n$

(C) There are $\frac{k!}{(k-r)!r!}$ subsets with cardinality $r \leq k$

(D) None of the above

Multiple Choice Question

Let S be an arbitrary set with cardinality n.
Suppose we have some $r \in \mathbb{N}$ such that $r \leq n$
Claim: The number of subsets of S with cardinality r is equal to

$$\frac{n!}{(n-r)!r!}$$

Inductive Step ($n > 0$).

I.H.: There are $\frac{k!}{(k-r)!r!}$ subsets with cardinality $r \leq k$.
Select an r-element subset, $R \subseteq S = \{s_1, s_2, \ldots, s_k, s_{k+1}\}$. $\square$

How do you think we should proceed?

(A) Proof by Induction on r
(B) Proof by Cases
(C) Try out some algebra
(D) None of the above

Multiple Choice Question

Claim: # subsets with cardinality r is $\frac{n!}{(n-r)!r!}$

Inductive Step ($n = k + 1 > 0$).

I.H.: There are $\frac{k!}{(k-r)!r!}$ subsets with cardinality $r \leq k$.
Select an r-element subset, $R \subseteq S = \{s_1, s_2, \ldots, s_k, s_{k+1}\}$.

Case: Suppose R `[has some property]`

Case: Suppose R `[has some other property]`

□

What two cases about the subset R might we consider?

(A) R is empty, R is not empty

(B) R contains s_{k+1}, R does not contain s_{k+1}

(C) R contains 0 elements, R contains > 0 elements

(D) R contains r elements, R contains $r + 1$ elements

Multiple Choice Question

Claim: # subsets with cardinality r is $\frac{n!}{(n-r)!r!}$
I.H.: There are $\frac{k!}{(k-r)!r!}$ subsets with cardinality $r \leq k$.

Inductive Step ($n = k + 1 > 0$), Case 1.

Select an r-element subset, $R \subseteq S = \{s_1, s_2, \ldots, s_k, s_{k+1}\}$.
Suppose $s_{k+1} \notin R$. Then $R \subseteq \{???\}$. □

If R can't contain s_{k+1}, what set of elements are we selecting from?

(A) S

(B) $S \setminus s_{k+1}$

(C) $\{s_1, s_2, \ldots, s_k\}$

(D) $\{s_2, s_3, \ldots, s_k, s_{k+1}\}$

Multiple Choice Question

Claim: # subsets with cardinality r is $\frac{n!}{(n-r)!r!}$
I.H.: There are $\frac{k!}{(k-r)!r!}$ subsets with cardinality $r \leq k$.

Inductive Step ($n = k + 1 > 0$), Case 1.

Select an r-element subset, $R \subseteq S = \{s_1, s_2, \ldots, s_k, s_{k+1}\}$.
Suppose $s_{k+1} \notin R$. Then $R \subseteq \{s_1, s_2, \ldots, s_k\}$, and there are ??? ways to select R. □

By the Inductive Hypothesis, how many ways are there to select an R that does not contain s_{k+1}?

(A) $n!/((n-r)!r!)$

(B) $k!/((k-r)!r!)$

(C) $k!/((k-(r-1))!(r-1)!)$

(D) We can't say for sure; maybe S had 0 elements to begin with

Multiple Choice Question

Claim: # subsets with cardinality r is $\frac{n!}{(n-r)!r!}$

I.H.: There are $\frac{k!}{(k-r)!r!}$ subsets with cardinality $r \leq k$.

Inductive Step ($n = k + 1 > 0$), Case 1.

Select an r-element subset, $R \subseteq S = \{s_1, s_2, \ldots, s_k, s_{k+1}\}$. Suppose $s_{k+1} \notin R$. Then $R \subseteq \{s_1, s_2, \ldots, s_k\}$, and there are

$$\frac{k!}{(k-r)!r!}$$

ways to select R, by the I.H. □

What can we conclude?

(A) There are $n!/((n-r)!r!)$ possible subsets of S

(B) There are $k!/((k-r)!r!)$ possible subsets of R

(C) When $|S| > 0$, there are $n!/((n-r)!r!)$ possible subsets of S

(D) None of the above

Multiple Choice Question

Claim: # subsets with cardinality r is $\frac{n!}{(n-r)!r!}$
I.H.: There are $\frac{k!}{(k-r)!r!}$ subsets with cardinality $r \leq k$.

Inductive Step $(n = k + 1 > 0)$, Case 2.

Select an r-element subset, $R \subseteq S = \{s_1, s_2, \ldots, s_k, s_{k+1}\}$.
Suppose $s_{k+1} \in R$. Then ???. $\square$

Which of the following is the most correct and useful description of R in this case?

(A) $R \subseteq \{s_1, s_2, \ldots, s_k, s_{k+1}\}$
(B) $R = \{s_1, s_2, \ldots, s_k, s_{k+1}\}$
(C) $R \subseteq S$
(D) $R = \{s_{k+1}\} \cup (R \setminus \{s_{k+1}\})$

Multiple Choice Question

Claim: # subsets with cardinality r is $\frac{n!}{(n-r)!r!}$
I.H.: There are $\frac{k!}{(k-r)!r!}$ subsets with cardinality $r \leq k$.

Inductive Step ($n = k + 1 > 0$), Case 2.

Select an r-element subset, $R \subseteq S = \{s_1, s_2, \ldots, s_k, s_{k+1}\}$.
Suppose $s_{k+1} \in R$. Then $R = \{s_{k+1}\} \cup (R \setminus \{s_{k+1}\})$. Thus, we can select R by selecting $R \setminus \{s_{k+1}\}$ and adding s_{k+1}. $\square$

How many elements are in $R \setminus \{s_{k+1}\}$?

(A) r

(B) k

(C) $r - 1$

(D) $k - 1$

Multiple Choice Question

Claim: $\#$ subsets with cardinality r is $\frac{n!}{(n-r)!r!}$
I.H.: There are $\frac{k!}{(k-r)!r!}$ subsets with cardinality $r \leq k$.

Inductive Step ($n = k + 1 > 0$), Case 2.

Select an r-element subset, $R \subseteq S = \{s_1, s_2, \ldots, s_k, s_{k+1}\}$.
Suppose $s_{k+1} \in R$. Then $R = \{s_{k+1}\} \cup (R \setminus \{s_{k+1}\})$. Thus, we can select R by selecting $R \setminus \{s_{k+1}\}$ and adding s_{k+1}. □

Consider selecting the members of $R \setminus \{r_{k+1}\} \subseteq S$. How many elements are we drawing from?

(A) r

(B) k

(C) $r - 1$

(D) $k - 1$

Multiple Choice Question

Claim: # subsets with cardinality r is $\frac{n!}{(n-r)!r!}$
I.H.: There are $\frac{k!}{(k-r)!r!}$ subsets with cardinality $r \leq k$.

Inductive Step ($n = k + 1 > 0$), Case 2.

Select an r-element subset, $R \subseteq S = \{s_1, s_2, \ldots, s_k, s_{k+1}\}$.
Suppose $s_{k+1} \in R$. Then $R = \{s_{k+1}\} \cup (R \setminus \{s_{k+1}\})$. Thus, we can select R by selecting $R \setminus \{s_{k+1}\}$ and adding s_{k+1}. There are ??? ways to do this. $\square$

By the I.H., how many ways can we select $R \setminus \{s_{k+1}\}$?

(A) $n!/((n-r)!r!)$

(B) $k!/((k-r)!r!)$

(C) $k!/((k-(r-1))!(r-1)!)$

(D) We can't say for sure; maybe R had 0 elements to begin with

Multiple Choice Question

Claim: # subsets with cardinality r is $\frac{n!}{(n-r)!r!}$

I.H.: There are $\frac{k!}{(k-r)!r!}$ subsets with cardinality $r \leq k$.

Inductive Step ($n = k + 1 > 0$).

Select an r-element subset, $R \subseteq S = \{s_1, s_2, \ldots, s_k, s_{k+1}\}$.

Case 1 ($s_{k+1} \notin R$): $\frac{k!}{(k-r)!r!}$

Case 2 ($s_{k+1} \in R$): $\frac{k!}{(k-(r-1))!(r-1)!}$

What can we conclude about the number of possible subsets when $n = k + 1$?

(A) It is the sum of the values in Cases 1 & 2

(B) It is the product of the values in Cases 1 & 2

(C) It might be the Case 1 value or it might be the Case 2 value

(D) None of the above

Algebra Question

Claim: # subsets with cardinality r is $\frac{n!}{(n-r)!r!}$

Inductive Step ($n = k + 1 > 0$).

Select an r-element subset, $R \subseteq S = \{s_1, s_2, \ldots, s_k, s_{k+1}\}$.
By the Principle of Addition, we have a total of

$$\frac{k!}{(k-r)!r!} + \frac{k!}{(k-(r-1))!(r-1)!}$$

$\square$

$(k-(r-1))! \quad = \quad ?$

Algebra Question

Claim: # subsets with cardinality r is $\frac{n!}{(n-r)!r!}$

Inductive Step ($n = k + 1 > 0$).

Select an r-element subset, $R \subseteq S = \{s_1, s_2, \ldots, s_k, s_{k+1}\}$.
By the Principle of Addition, we have a total of

$$\frac{(k-r+1)}{(k-r+1)} \cdot \frac{k!}{(k-r)!r!} + \frac{k!}{(k-r+1)!(r-1)!} \cdot \frac{r}{r}$$

$\square$

$(k - r + 1) \cdot (k - r)! \quad = \quad ?$

Algebra Question

Claim: # subsets with cardinality r is $\frac{n!}{(n-r)!r!}$

Inductive Step $(n = k + 1 > 0)$.

Select an r-element subset, $R \subseteq S = \{s_1, s_2, \ldots, s_k, s_{k+1}\}$.
By the Principle of Addition, we have a total of

$$\frac{k! \cdot (k-r+1)}{(k-r+1)!r!} + \frac{k!}{(k-r+1)!(r-1)!} \cdot \frac{r}{r}$$

$\square$

$(r-1)! \cdot r \quad = \quad ?$

Algebra Question

Claim: # subsets with cardinality r is $\frac{n!}{(n-r)!r!}$

Inductive Step $(n = k + 1 > 0)$.

Select an r-element subset, $R \subseteq S = \{s_1, s_2, \ldots, s_k, s_{k+1}\}$.
By the Principle of Addition, we have a total of

$$\frac{k! \cdot (k - r + 1)}{(k - r + 1)!r!} + \frac{k! \cdot r}{(k - r + 1)!r!}$$

□

$$k! \cdot (k - r + 1) + k! \cdot r \quad = \quad ?$$

Algebra Question

Claim: # subsets with cardinality r is $\frac{n!}{(n-r)!r!}$

Inductive Step $(n = k + 1 > 0)$.

Select an r-element subset, $R \subseteq S = \{s_1, s_2, \ldots, s_k, s_{k+1}\}$.
By the Principle of Addition, we have a total of

$$\frac{k! \cdot (k - r + 1 + r)}{(k - r + 1)!r!}$$

$\square$

$(k - r + 1 + r) \quad = \quad ?$

Algebra Question

Claim: # subsets with cardinality r is $\frac{n!}{(n-r)!r!}$

Inductive Step ($n = k + 1 > 0$).

Select an r-element subset, $R \subseteq S = \{s_1, s_2, \ldots, s_k, s_{k+1}\}$.
By the Principle of Addition, we have a total of

$$\frac{k! \cdot (k+1)}{(k-r+1)!r!}$$

$\square$

$k! \cdot (k+1) \quad = \quad ?$

Remainder of Proof

Claim: # subsets with cardinality r is $\frac{n!}{(n-r)!r!}$

Inductive Step ($n = k + 1 > 0$).

Select an r-element subset, $R \subseteq S = \{s_1, s_2, \ldots, s_k, s_{k+1}\}$.
By the Principle of Addition, we have a total of

$$\frac{(k+1)!}{((k+1)-r)!r!}$$

ways of selecting R. $\square$

Multiple Choice Question

Let S be an arbitrary set with cardinality n.
Suppose we have some $r \in \mathbb{N}$ such that $r \leq n$
When we selected a subset $R \subseteq S$ with cardinality r, did the order of the elements matter?

(A) Yes

(B) No

Combination

Definition

A combination is a way of selecting several things out of a larger group where order does not matter.

The standard definition aligns with what we've looked at: the number of ways to select an r-element subset of an n-element set.

Definition (Notation)

There are many notations for the number of combinations of r distinct objects chosen from n distinct objects:

- $C(n, r)$
- ${}_nC_r$
- C_r^n
- $\binom{n}{r}$

Generally pronounced "n choose r".

Multiple Choice Question

There are 33 students enrolled in this class.
I need to pick 5 students for some stupid in-class demo.
How many ways can I select the group?

(A) $C(5, 27)$
(B) $C(27, 5)$
(C) $C(5, 33)$
(D) None of the above

Multiple Choice Question

A standard deck of playing cards has 52 cards.
A poker hand consists of 5 cards (in 5-card draw, anyway).
How many different hands are possible?

(A) $C(5, 52)$

(B) $C(52, 5)$

(C) $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$

(D) None of the above

Multiple Choice Question

How many different 7-digit phone numbers are there?

(A) $C(10, 7)$

(B) $C(7, 10)$

(C) $C(10, 3)$

(D) None of the above

Multiple Choice Question

How many different 7-digit phone numbers are there with no duplicate digits?

(A) $C(10, 7)$

(B) $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$

(C) $10 + 9 + 8 + 7 + 6 + 5 + 4$

(D) None of the above

Permutation

Definition

A permutation is an ordered arrangement of objects.
We count the number of permutations of r distinct objects selected from a set of n distinct objects, and use any of the following notations:

- $P(n, r)$
- ${}_nP_r$
- P_r^n

Multiple Choice Question

Suppose we select a combination of the 4-element set

$$\{a, b, c, d\}$$

How many different 3-combinations are there?

(A) 2

(B) 3

(C) 8

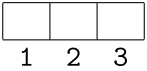
(D) None of the above

Multiple Choice Question

Consider the possible 3-combinations of the 4-element set $\{a, b, c, d\}$:

- $\{a, b, c\}$ $\longleftarrow$
- $\{a, b, d\}$
- $\{a, c, d\}$
- $\{b, c, d\}$

Let's think about how many ways there are to order the first combination.



How many choices do we have for the position of a?

(A) 1

(B) 2

(C) 3

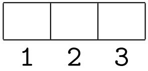
(D) None of the above

Multiple Choice Question

Consider the possible 3-combinations of the 4-element set $\{a, b, c, d\}$:

- $\{a, b, c\} \longleftarrow$
- $\{a, b, d\}$
- $\{a, c, d\}$
- $\{b, c, d\}$

Let's think about how many ways there are to order the first combination.



After placing a, how many choices do we have for the position of b?

(A) 1

(B) 2

(C) 3

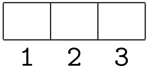
(D) None of the above

Multiple Choice Question

Consider the possible 3-combinations of the 4-element set $\{a, b, c, d\}$:

- $\{a, b, c\} \longleftarrow$
- $\{a, b, d\}$
- $\{a, c, d\}$
- $\{b, c, d\}$

Let's think about how many ways there are to order the first combination.



After placing a & b, how many choices do we have for the position of c?

(A) 1

(B) 2

(C) 3

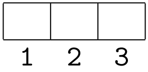
(D) None of the above

Multiple Choice Question

Consider the possible 3-combinations of the 4-element set $\{a, b, c, d\}$:

- $\{a, b, c\} \longleftarrow$
- $\{a, b, d\}$
- $\{a, c, d\}$
- $\{b, c, d\}$

Let's think about how many ways there are to order the first combination.



3 ??? 2 ??? 1

What rule do we use to combine these choices into the total number of orderings of the first combination?

(A) The Principle of Addition

(B) The Principle of Multiplication

(C) $C(n, r)$

(D) $P(n, r)$

Multiple Choice Question

Consider the possible 3-combinations of the 4-element set $\{a, b, c, d\}$:

- $\{a, b, c\} \longleftarrow$ 6 orderings
- $\{a, b, d\} \longleftarrow$?
- $\{a, c, d\}$
- $\{b, c, d\}$

How many orderings do you suppose there are of the second combination?

(A) 3

(B) 6

(C) 9

(D) None of the above

Multiple Choice Question

Consider the possible 3-combinations of the 4-element set $\{a, b, c, d\}$:

- $\{a, b, c\} \longleftarrow$ 6 orderings
- $\{a, b, d\} \longleftarrow$ 6 orderings
- $\{a, c, d\} \longleftarrow$?
- $\{b, c, d\}$

How many orderings do you suppose there are of the third combination?

(A) 3

(B) 6

(C) 9

(D) None of the above

Multiple Choice Question

Consider the possible 3-combinations of the 4-element set $\{a, b, c, d\}$:

- $\{a, b, c\} \longleftarrow$ 6 orderings
- $\{a, b, d\} \longleftarrow$ 6 orderings
- $\{a, c, d\} \longleftarrow$ 6 orderings
- $\{b, c, d\} \longleftarrow$ 6 orderings

What is the total number of possible orderings of 3 arbitrary elements selected from the original set of 4?

(A) $6 \times 6 \times 6 \times 6$

(B) $6 + 6 + 6 + 6$

(C) $(6 + 6) \times (6 + 6)$

(D) $6 \times 5 \times 4 \times 3$

Permutations and Combinations

Theorem

The number of permutations of r distinct objects selected from n distinct objects is

$$\begin{aligned} P(n,r) &= C(n,r)\cdot r! \\ &= \frac{n!}{(n-r)!r!}\cdot r! \\ &= \frac{n!}{(n-r)!} \\ &= n \times (n-1) \times (n-2) \times \cdots \times (n-r+1) \end{aligned}$$

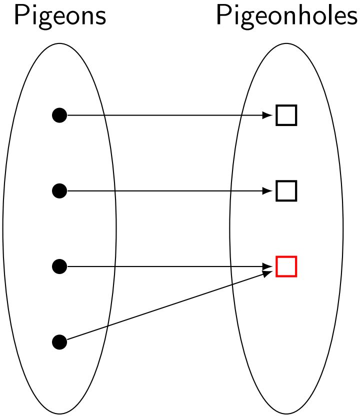
Theorem (Corollary)

$$C(n,r) = \frac{P(n,r)}{r!}$$

The Pigeonhole Principle

Definition (Simple)

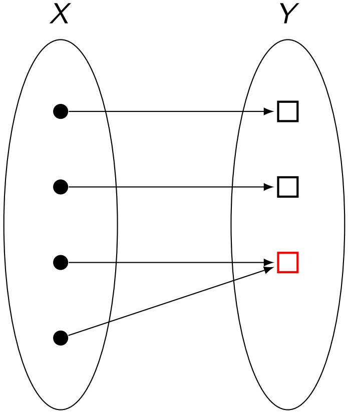
If you put $n+1$ pigeons into n pigeonholes, then at least one hole contains two or more pigeons.



The Pigeonhole Principle

Definition (General)

Let X and Y be finite sets such that $|X| > |Y|$. Then there cannot exist a 1-1 mapping from X to Y.



{One, Many}-to-{One, Many}

The Pigeonhole Principle In Action

Hint: always ask yourself:

- What are the pigeons?
- What are the pigeonholes?

Example

I have 6 white socks and 6 black socks placed loosely in a drawer. How many socks must I pull out at random to guarantee that I have a matched pair of socks?

(A) 2

(B) 3

(C) 4

(D) Depends on the number of socks you have to begin with

The Pigeonhole Principle In Action

Hint: always ask yourself:

- What are the pigeons?
- What are the pigeonholes?

Example

At least two people in this classroom have birthdays in the same month.

(A) True

(B) False

(C) I don't know

The Pigeonhole Principle In Action

Hint: always ask yourself:

- What are the pigeons?
- What are the pigeonholes?

Example

At least two people in this classroom have birthdays on the same day of the month (1^{st}, 2^{nd}, 3^{rd}, etc.).

(A) True

(B) False

(C) I don't know

The Pigeonhole Principle In Action

Hint: always ask yourself:

- What are the pigeons?
- What are the pigeonholes?

Example

The average adult human head has about $150,000$ hairs.
There must be two people in this classroom with the same number of hairs on their heads.

(A) True

(B) False

(C) I don't know

The Pigeonhole Principle In Action

Hint: always ask yourself:

- What are the pigeons?
- What are the pigeonholes?

Example

The average adult human head has about $150,000$ hairs.
There must be two people in the state of California with the same number of hairs on their heads.

(A) True

(B) False

(C) I don't know

Multiple Choice Question

Note

Let's assume friendship is a two-way relationship. If person A says person B is their friend, then person B must say that person A is their friend.

Claim: In any group of people, there are at least two people with the same number of friends from that group.
What should the pigeons be?

(A) People
(B) Number of friends in the group
(C) Friendships
(D) None of the above

Multiple Choice Question

Note

Let's assume friendship is a two-way relationship. If person A says person B is their friend, then person B must say that person A is their friend.

Claim: In any group of people, there are at least two people with the same number of friends from that group.

What should the pigeonholes be?

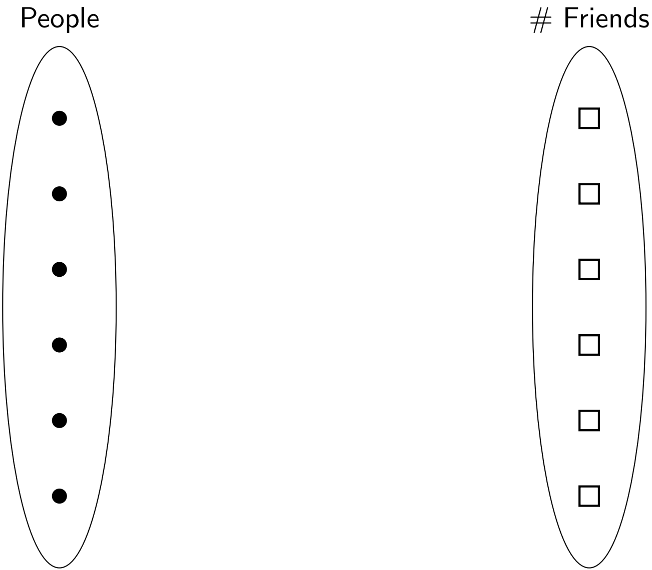
(A) People

(B) Number of friends in the group

(C) Friendships

(D) None of the above

Visualizing



Multiple Choice Question

Let's take our 6-person group as an example.
Suppose there's one person who has 0 friends in the group.
That being the case, is it possible for anyone else to have $n - 1$ (e.g., 5) friends in the group?

(A) Yes

(B) No

(C) I don't know

Multiple Choice Question

Let's take our 6-person group as an example.
Suppose there's one person who has $n-1$ (e.g., 5) friends in the group.
That being the case, is it possible for anyone else to have 0 friends in the group?

(A) Yes

(B) No

(C) I don't know

Multiple Choice Question

Thus, we must either

- Fit n pigeons into the pigeonholes $\{1, 2, 3, \ldots, n-1\}$, of which there are $n-1$.
- Fit n pigeons into the pigeonholes $\{0, 1, 2, \ldots, n-2\}$, of which there are $n-1$.

Thus, when someone says "0" or "$n-1$", we'll have two people with the same number of friends.

What if nobody says they have 0 or $n-1$ friends?

(A) Impossible

(B) Doesn't matter—there would be $n-2$ pigeonholes, in that case

(C) The property doesn't actually hold then

(D) This violates the 2-way definition of "friendship" we had

Multiple Choice Question

Even with a realistic definition of "friendship", there are interesting properties.

Theorem (On Friends And Strangers)

In a group of 6 people, there are guaranteed to be at least 3 mutual friends or 3 mutual strangers.

What would the pigeons be?

(A) People

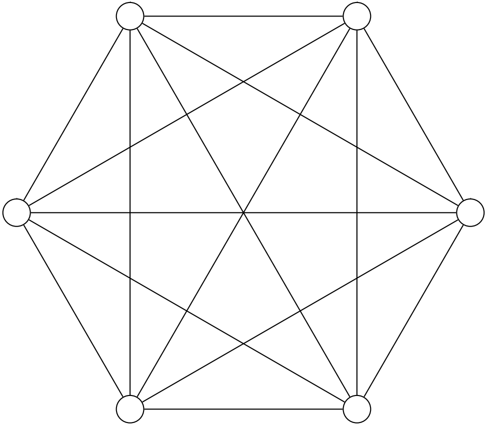
(B) Friendships

(C) Number of friends in the group

(D) I don't know / None of the above

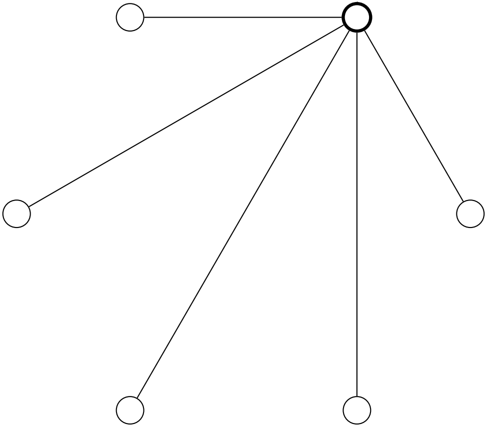
Graph

Consider the relationships between six people. . .



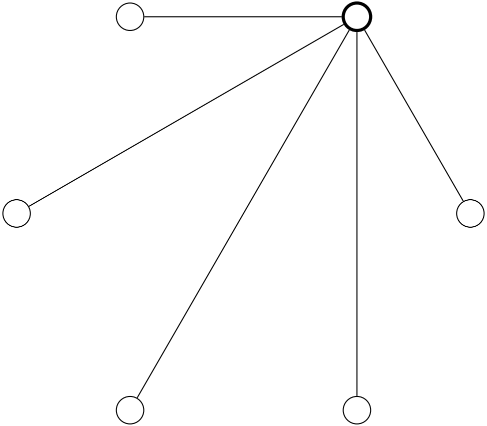
Graph

Consider an arbitrary person. . .



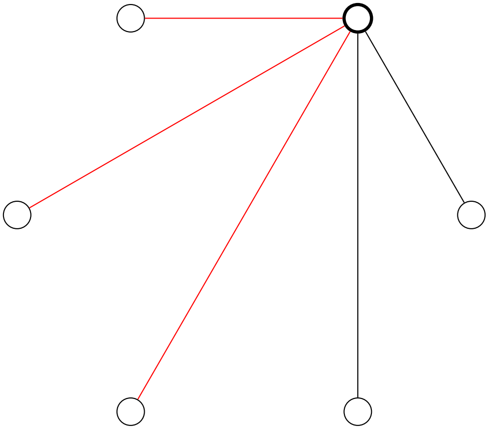
Graph

The remaining $\overbrace{\text{5 people}}^{\text{pigeons}}$ are either $\overbrace{\text{friends or strangers}}^{\text{pigeonholes}}$...



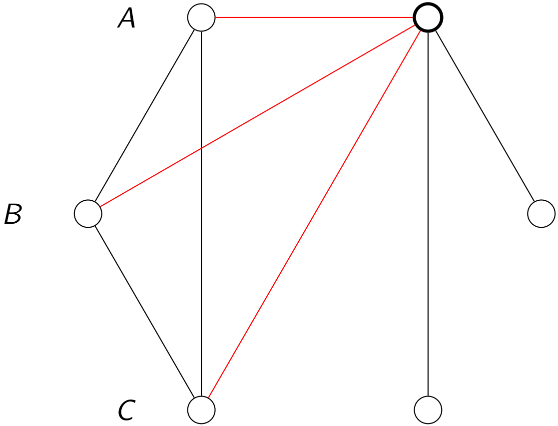
Graph

There $\overbrace{\text{must be 3 of the same color}}^{\text{Pigeonhole Principle}}$. WLOG, let's say there are 3 friends...



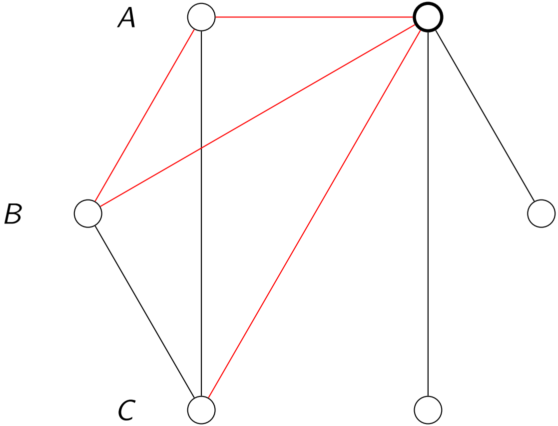
Graph

Consider the relationships between those 3...



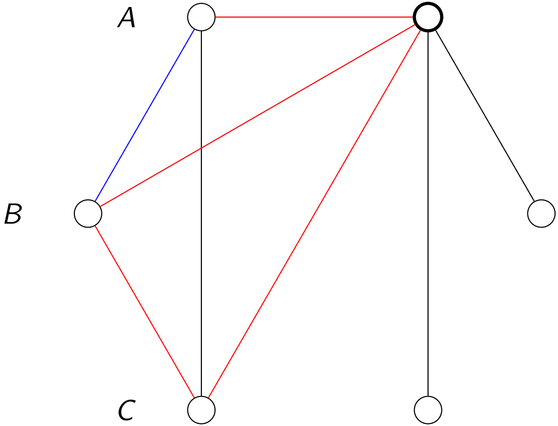
Graph

Case 1: A and B are friends...



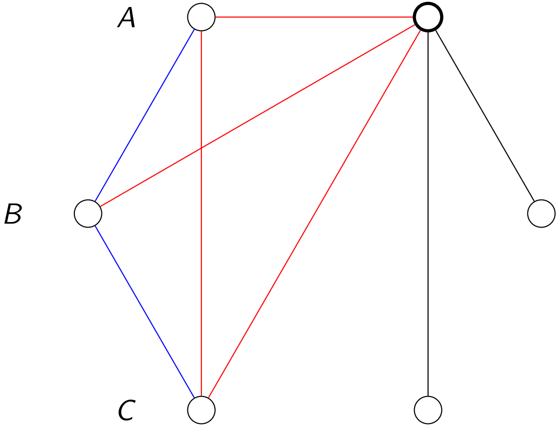
Graph

Case 2: B and C are friends...



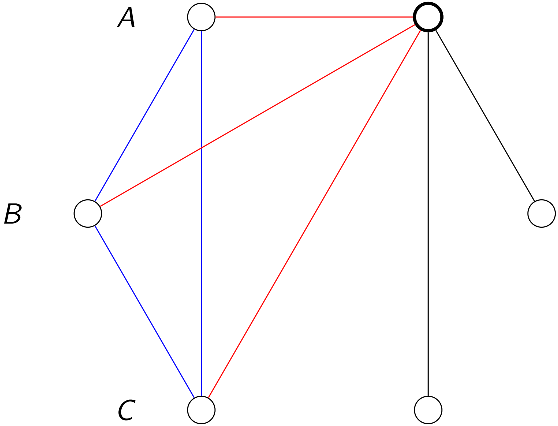
Graph

Case 3: A and C are friends. . .



Graph

Case 4: A, B, and C aren't friends...



Counting With Sets

Multiple Choice Question

With arbitrary sets A and B, what do you suppose is the value of the following?

$$|A \cap B|$$

(A) $|A| + |B|$
(B) $|A| \times |B|$
(C) $|A| - |B|$
(D) Can't say

With arbitrary sets A and B, what do you suppose is the value of the following?

$$|A \setminus B|$$

(A) $|A| - |B|$
(B) $|A| - |A \cap B|$
(C) $|A| - |A \cup B|$
(D) None of the above

With arbitrary sets A and B, what do you suppose is the value of the following?

$$(A \setminus B) \cap (B \setminus A)$$

(A) $\{\ \}$
(B) $A \cup B$
(C) $A \cap B$
(D) $A \,\Delta\, B$

With arbitrary sets A and B, what do you suppose is the value of the following?

$$(A \setminus B) \cap (A \cap B)$$

(A) $\{ \}$
(B) $A \cup B$
(C) $A \cap B$
(D) $B \setminus A$

Disjoint Sets

Definition

Sets A and B are disjoint if $A \cap B = \{ \}$.

Definition

The disjoint union of sets A and B is defined simply as their union.
We just use the notation $A \uplus B$ to underscore the fact that A and B are disjoint.

Multiple Choice Question

With arbitrary sets A and B, what do you suppose is the value of the following?

$$|A \cup B|$$

(A) $|A| + |B|$
(B) $|A| + |A \cap B|$
(C) Impossible to say
(D) None of the above

Principle of Inclusion/Exclusion

Theorem

For any two arbitrary finite sets, A and B,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Theorem (Corollary)

For arbitrary finite sets A and B,

$$|A \uplus B| = ???$$

(A) $|A| + |B| - |A \cap B|$
(B) $|A \setminus B| + |B \setminus A|$
(C) $|A| + |B|$
(D) $|A \cup B|$

Principle of Inclusion/Exclusion

Theorem

For any two arbitrary finite sets, A and B,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Theorem (Corollary)

For arbitrary finite sets A and B,

$$|A \uplus B| = |A| + |B|$$

(A) $|A| + |B| - |A \cap B|$

(B) $|A \setminus B| + |B \setminus A|$

(C) $|A| + |B|$

(D) $|A \cup B|$

Principle of Inclusion/Exclusion

Proof.

Notice that $A \cup B = (A \setminus B) \uplus (A \cap B) \uplus (B \setminus A)$. Then,

$$\begin{aligned} |A \cup B| &= |(A \setminus B) \uplus (A \cap B) \uplus (B \setminus A)| \\ &= |A \setminus B| + |A \cap B| + |B \setminus A| \\ &= |A| - |A \cap B| + |A \cap B| + |B| - |A \cap B| \\ &= |A| + |B| - |A \cap B| \end{aligned}$$

$\square$

Multiple Choice Question

Suppose you want to order a dessert—either a cake or some ice cream. Specifically, you have the following choices.

$$\begin{aligned} \text{Cake} &: \text{chocolate}, \ \text{angel's food}, \ \text{carrot} \\ \text{Ice Cream} &: \text{vanilla}, \ \text{strawberry}, \ \text{pistachio} \end{aligned}$$

How many ways can you select a dessert?

(A) 1

(B) 3

(C) 6

(D) 9

Multiple Choice Question

Suppose you want to order a dessert—either a cake or some ice cream. Specifically, you have the following choices.

$$\text{Cake} = \{\text{chocolate},\ \text{angel's food},\ \text{carrot}\}$$
$$\text{Ice Cream} = \{\text{vanilla},\ \text{strawberry},\ \text{pistachio}\}$$

Are “Cake” and “Ice Cream” disjoint sets?

(A) Yes

(B) No

(C) I don't know

(D) Maybe

Multiple Choice Question

Suppose you want to order a dessert—either a cake or some ice cream. Specifically, you have the following choices.

$$\begin{aligned} \text{Cake} &= \{\text{chocolate},\ \text{angel's food},\ \text{carrot}\} \\ \text{Ice Cream} &= \{\text{vanilla},\ \text{strawberry},\ \text{pistachio}\} \end{aligned}$$

What is $|\text{Cake} \uplus \text{Ice Cream}|$?

(A) 1

(B) 3

(C) 6

(D) 9

Multiple Choice Question

Suppose you want to buy an automobile. You go see the selection available from two dealers, and find the dealers had the following cars:

$$\text{Dealer } 1 = \{\text{Porsche Panamera}, \text{ Cadillac Escalade}, \text{ Honda Civic}\}$$
$$\text{Dealer } 2 = \{\text{Ford Thundercougarfalconbird}, \text{ Honda Civic}, \text{ Mini-Cooper}\}$$

How many choices do you have for your car model?

(A) $|\text{Dealer } 1| + |\text{Dealer } 2|$

(B) $|\text{Dealer } 1| + |\text{Dealer } 2| - |\text{Dealer } 1 \cup \text{Dealer } 2|$

(C) $|\text{Dealer } 1| + |\text{Dealer } 2| - |\text{Dealer } 1 \cap \text{Dealer } 2|$

(D) None of the above

Multiple Choice Question

Suppose we have two sets representing events:

$$A = \text{the possible outcomes of the first event}$$
$$B = \text{the possible outcomes of the second event}$$

The Addition Principle says that $|A \uplus B| = |A| + |B|$.
What do you suppose the Multiplication Principle says?

(A) $|A \cup B| = |A| + |B| - |A \cap B|$

(B) $|A \cap B| = |A| \times |B|$

(C) $|A \times B| = |A| \times |B| - |A \cap B|$

(D) $|A \times B| = |A| \times |B|$