Induction

CS 130

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Peano Arithmetic

Definitions (Axioms)

1. 0 is a natural number
2. For every natural number x, $x = x$
3. For any two natural numbers x and y, if $x = y$ then $y = x$
4. For any three natural numbers x, y, and z, if $x = y$ and $y = z$, then $x = z$
5. For any objects a and b, if a is a natural number and $a = b$, then b is a natural number
6. For every natural number n, n' is also a natural number
7. For every natural number n, $n' = 0$ is false
8. For all natural numbers m and n, if $m' = n'$ then $m = n$

★9. Principle of Induction (we'll cover this now!)

The Principle of Induction

Definition

Let P be a predicate upon natural numbers. If both of the following hold, then the predicate P must be true of every possible natural number

1. $P(0)$ is true **(Basis)**

AND 2. $P(k) \implies P(k+1)$ is true for every natural number k **(Inductive Step)**

$P(k)$ is often called the **Inductive Hypothesis** (or **I.H.**, for short)

Formally,

$$P(0), \quad \forall k[(P(k) \implies P(k'))] \qquad \vdash \qquad \forall n[P(n)]$$

Multiple Choice Question

How can we generally prove the implication $P(k) \implies P(k+1)$ is true?

(A) Assume $P(k)$ is true, show $P(k+1)$ must be true

(B) Assume $\neg P(k)$ is true, show $\neg P(k+1)$ must be true

(C) Assume $\neg P(k+1)$ is true, show $P(k)$ must be true

(D) None of the above

Multiple Choice Question

Supose we have the predicate $P(n) =$ "n is even".
$P(0)$ is true, since 0 is even.

Does $P(n)$ hold for every possible n?

(A) Yes

(B) No

Multiple Choice Question

Suppose we have the predicate $P(n) =$ "$n = n + 3$".
Assume $P(k)$ is true—that is, $k = k + 3$. Then,

$$k = k + 3 \qquad \text{(Assumption from above)}$$
$$(k + 1) = (k + 1) + 3 \qquad \text{(Add 1 to both sides)}$$

That is, $P(k + 1)$ is also true.
$\therefore P(k) \implies P(k + 1)$ for every natural number k.

Does $P(n)$ hold for every possible natural number n?

(A) Yes

(B) No

Multiple Choice Question

In general, suppose $\forall k[(P(k) \implies P(k+1))]$. Intuitively, this means that

$$\begin{aligned} & (P(0) \implies P(1)) \\ \wedge\ & (P(1) \implies P(2)) \\ \wedge\ & (P(2) \implies P(3)) \\ & \vdots \end{aligned}$$

With just this information, can we deduce that $P(1)$ must be true?

(A) Yes—we know $(P(0) \implies P(1))$ is true

(B) Yes—we know $(P(1) \implies P(2))$ is true

(C) No—$(P(0) \implies P(1))$ could still be true if $P(1)$ were false

(D) No—$(P(1) \implies P(2))$ could still be true if $P(1)$ were false

Multiple Choice Question

In general, suppose $\forall k[(P(k) \implies P(k+1))]$. Intuitively, this means that

$$\begin{aligned} & (P(0) \implies P(1)) \\ \wedge\ & (P(1) \implies P(2)) \\ \wedge\ & (P(2) \implies P(3)) \\ & \vdots \end{aligned}$$

Suppose we also know that $P(0)$ is true.
With just this information, can we deduce that $P(1)$ must be true?

(A) Yes

(B) No

Multiple Choice Question

In general, suppose $\forall k[(P(k) \implies P(k+1))]$. Intuitively, this means that

$$
\begin{aligned}
& (P(0) \implies P(1)) \\
\wedge\ & (P(1) \implies P(2)) \\
\wedge\ & (P(2) \implies P(3)) \\
& \vdots
\end{aligned}
$$

Suppose we also know that $P(0)$ is true.
With just this information, can we deduce that $P(2)$ must be true?

(A) Yes

(B) No

Multiple Choice Question

In general, suppose $\forall k[(P(k) \implies P(k+1))]$. Intuitively, this means that

$$
\begin{aligned}
& (P(0) \implies P(1)) \\
\wedge\ & (P(1) \implies P(2)) \\
\wedge\ & (P(2) \implies P(3)) \\
& \vdots
\end{aligned}
$$

Suppose we also know that $P(0)$ is true.
With just this information, can we deduce that $P(3)$ must be true?

(A) Yes

(B) No

Multiple Choice Question

Let's consider how to prove the following equality is true for any natural number, n.

$$P(n) = \boxed{\sum_{i=0}^{n} i \quad = \quad \frac{n(n+1)}{2}}$$

What is the basis?

(A) $\sum_{i=0}^{0} i$

(B) $\sum_{i=0}^{n} i = 0$

(C) $\sum_{i=0}^{0} i = 0(0+1)/2$

(D) $\sum_{i=0}^{n} i = 0(0+1)/2$

Multiple Choice Question

Let's consider how to prove the following equality is true for any natural number, n.

$$P(n) = \boxed{\sum_{i=0}^{n} i \quad = \quad \frac{n(n+1)}{2}}$$

Basis: $\sum_{i=0}^{0} = 0 \quad = \quad 0(0+1)/2 = 0$

Inductive Step: ...

What do we assume as the inductive hypothesis, $P(k)$?

(A) $k = k(k+1)/2$

(B) $\sum_{i=0}^{k} i = 0$

(C) $\sum_{i=0}^{n} i = k(k+1)/2$

(D) $\sum_{i=0}^{k} i = k(k+1)/2$

Multiple Choice Question

Let's consider how to prove the following equality is true for any natural number, n.

$$P(n) = \boxed{\sum_{i=0}^{n} i \quad = \quad \frac{n(n+1)}{2}}$$

Basis: $\sum_{i=0}^{0} = 0 \quad = \quad 0(0+1)/2 = 0$

Inductive Step:

$$\sum_{i=0}^{k} i = \frac{k(k+1)}{2} \qquad \text{(I.H.)}$$

What should we do to this equation to move towards proving $P(k+1)$?

(A) Add k to both sides

(B) Add $k+1$ to both sides

(C) Expand the definition of $\sum$

(D) Proof by Cases on whether $k = 0$ or $k > 0$

Multiple Choice Question

Basis: $\sum_{i=0}^{0} = 0 \quad = \quad 0(0+1)/2 = 0$

Inductive Step:

$$\sum_{i=0}^{k} i = \frac{k(k+1)}{2} \qquad \text{(I.H.)}$$

$$(k+1) + \sum_{i=0}^{k} i = (k+1) + \frac{k(k+1)}{2}$$

What is the left side of the equation equal to?

(A) $(k+1) + k + \sum_{i=0}^{k-1} i$

(B) $(k+1) + 0$

(C) $\sum_{i=0}^{k+1} i$

(D) Either (A) or (B)

Multiple Choice Question

Basis: $\sum_{i=0}^{0} = 0 \quad = \quad 0(0+1)/2 = 0$

Inductive Step:

$$\sum_{i=0}^{k} i = \frac{k(k+1)}{2} \qquad \text{(I.H.)}$$

$$(k+1) + \sum_{i=0}^{k} i = (k+1) + \frac{k(k+1)}{2}$$

$$\sum_{i=0}^{k+1} i = \frac{k^2+3k+2}{2}$$

$k^2 + 3k + 2 = \ldots ?$

(A) $(k-1)(k-2)$
(B) $(k+1)(k+2)$
(C) $(k+1)(k-2)$
(D) $(k-1)(k+2)$

Our First Inductive Proof

Basis: $\sum_{i=0}^{0} = 0 \quad = \quad 0(0+1)/2 = 0$

Inductive Step:

$$\sum_{i=0}^{k} i = \frac{k(k+1)}{2} \qquad \text{(I.H.)}$$

$$(k+1) + \sum_{i=0}^{k} i = (k+1) + \frac{k(k+1)}{2}$$

$$\sum_{i=0}^{k+1} i = \frac{(k+1)((k+1)+1)}{2}$$

I.e., assuming $P(k)$, we can show $P(k+1)$ is true

$\therefore \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$ for every natural number n

Multiple Choice Question

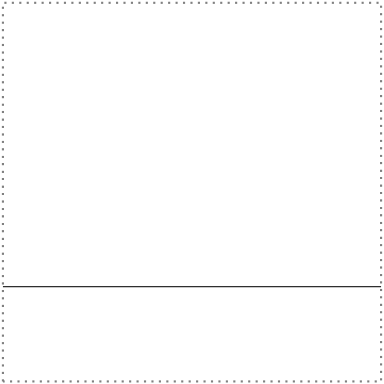
Picture an infinite region. Draw lines to divide it up (like slicing a pizza).

How many regions are there with 0 lines?

(A) 0 (B) 1 (C) 2 (D) 3

Multiple Choice Question

Picture an infinite region. Draw lines to divide it up (like slicing a pizza).

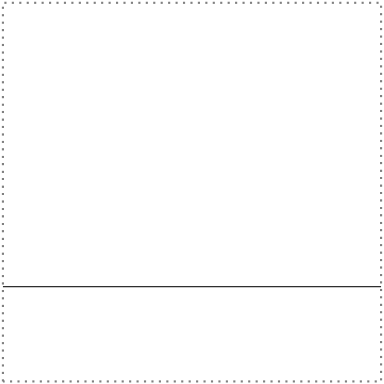


How many regions are there with 1 line?

(A) 0 (B) 1 (C) 2 (D) 3

Multiple Choice Question

Picture an infinite region. Draw lines to divide it up (like slicing a pizza).

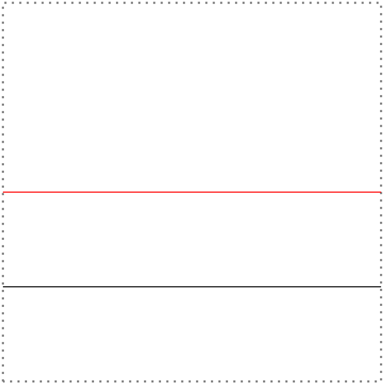


How many regions are there with 2 lines?

(A) 2 (B) 3 (C) 4 (D) Depends on how we draw

Multiple Choice Question

Picture an infinite region. Draw lines to divide it up (like slicing a pizza).

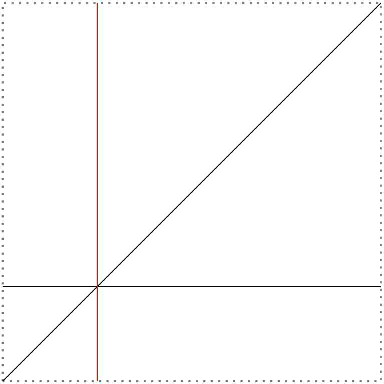


What is the maximum number of regions with 2 lines?

(A) 2 (B) 3 (C) 4 (D) None of these

Multiple Choice Question

Picture an infinite region. Draw lines to divide it up (like slicing a pizza).

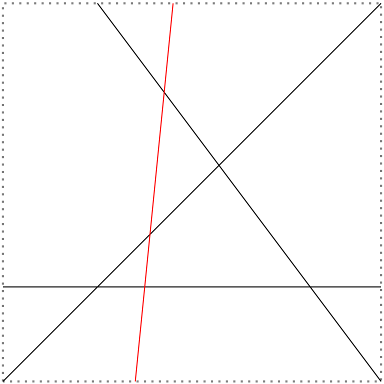


What is the maximum number of regions with 3 lines?

(A) 6 (B) 7 (C) 8 (D) None of these

Multiple Choice Question

Picture an infinite region. Draw lines to divide it up (like slicing a pizza).

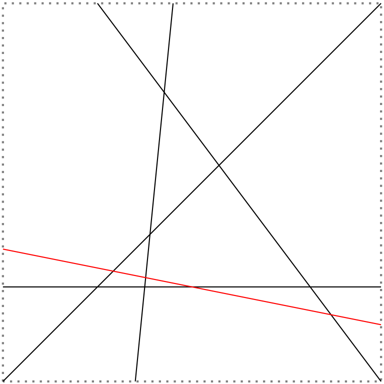


What is the maximum number of regions with 4 lines?

(A) 8 (B) 9 (C) 10 (D) None of these

Multiple Choice Question

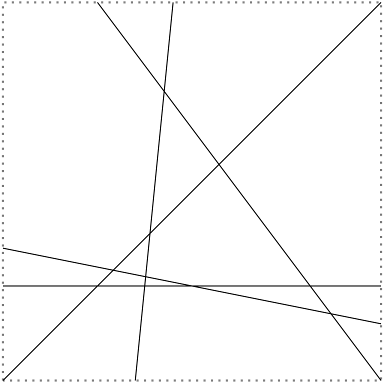
Picture an infinite region. Draw lines to divide it up (like slicing a pizza).



What is the maximum number of regions with 5 lines?

(A) 14 (B) 15 (C) 16 (D) None of these

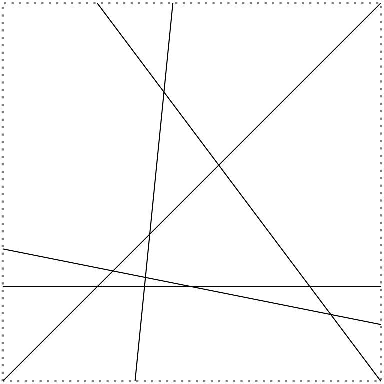
Multiple Choice Question



If we were to draw a sixth line to maximize the number of regions, how many other lines must it intersect?

(A) 2 (B) 3 (C) 4 (D) 5

Multiple Choice Question



How many more regions would the sixth line introduce to the above?

(A) 5 (B) 6 (C) 7 (D) Something more complex

Multiple Choice Question

On an infinite plane, draw $n \geq 0$ lines such that

- No two lines are parallel
- No three lines meet at a single common point

Claim: the number of regions this gives us is

$$\frac{n^2 + n + 2}{2}$$

Let's prove this by induction. What must we do first?

(A) Assume $P(k)$ is true, show $P(k+1)$

(B) Assume $P(0)$ is true

(C) Show $P(0)$ is true

(D) None of the above (we don't know what P is!)

Multiple Choice Question

Proof by Induction (# regions $= \frac{n^2+n+2}{2}$).

Basis: When we have 0 lines, we have 1 region, which $= \frac{0^2+0+2}{2}$

Inductive Step: ...

□

How do we begin the Inductive Step?

(A) Assume the Inductive Hypothesis

(B) Assume the claim holds at $n = k$

(C) Assume $P(k)$

(D) All of the above

Multiple Choice Question

Proof by Induction (# regions $= \frac{n^2+n+2}{2}$).

Basis: When we have 0 lines, we have 1 region, which $= \frac{0^2+0+2}{2}$

Inductive Step: Assume as our Inductive Hypothesis that k lines divide the plane into $\frac{k^2+k+2}{2}$ regions.
Drawing the $(k+1)^{\text{th}}$ line adds ??? more regions.

□

(A) k

(B) $k+1$

(C) k^2+k+2

(D) None of the above

Another Inductive Proof

Proof by Induction (# regions $= \frac{n^2+n+2}{2}$).

Basis: When we have 0 lines, we have 1 region, which $= \frac{0^2+0+2}{2}$

Inductive Step: Assume as our Inductive Hypothesis that k lines divide the plane into $\frac{k^2+k+2}{2}$ regions.
Drawing the $(k+1)^{\text{th}}$ line adds $k+1$ more regions.

$$\begin{aligned}\frac{k^2+k+2}{2}+(k+1) &= \frac{k^2+k+2+(2k+2)}{2}\\ &= \frac{(k^2+2k+1)+(k+1)+2}{2}\\ &= \frac{(k+1)^2+(k+1)+2}{2}\end{aligned}$$

□