Informal Proofs

CS 130

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Multiple Choice Question

Is the following WFF logically valid?

$$\forall x[(P(x) \vee \neg P(x))]$$

(A) Yes

(B) No

Multiple Choice Question

Is the following WFF logically valid?

$$\forall x[((P(x) \wedge B(x)) \implies O(x))]$$

(A) Yes

(B) No

Multiple Choice Question

Let the universe be the natural numbers $(0, 1, 2, \ldots)$. Also, let

$$P(x) = \text{“}x \text{ is prime”}$$
$$B(x) = \text{“}x > 2\text{”}$$
$$O(x) = \text{“}x \text{ is odd”}$$

Is the following WFF true?

$$\forall x[((P(x) \land B(x)) \implies O(x))]$$

(A) Yes
(B) No

Multiple Choice Question

Let the universe be the natural numbers $(0, 1, 2, \ldots)$. Also, let

$$P(x) = \text{"}x \text{ is prime"}$$
$$B(x) = \text{"}x > 2\text{"}$$
$$O(x) = \text{"}x \text{ is odd"}$$

Suppose we want to prove

$$\vdash \quad \forall x[((P(x) \wedge B(x)) \implies O(x))]$$

What does $P(x)$ mean in more "primitive" terms?

(A) $\forall y[((y > 1 \wedge y < x) \implies \neg\exists z[x = yz])]$

(B) $\exists y[x = 2y + 1]$

(C) $\exists y[x = 2y]$

(D) $O(x)$ must be true

Multiple Choice Question

Let the universe be the natural numbers $(0, 1, 2, \ldots)$. Also, let

$$P(x) = \text{“}x \text{ is prime”}$$
$$B(x) = \text{“}x > 2\text{”}$$
$$O(x) = \text{“}x \text{ is odd”}$$

Suppose we want to prove

$$\vdash \quad \forall x[((P(x) \land B(x)) \implies O(x))]$$

What does $O(x)$ mean in more “primitive” terms?

(A) $\forall y[((y > 1 \land y < x) \implies \neg\exists z[x = yz])]$

(B) $\exists y[x = 2y + 1]$

(C) $\exists y[x = 2y]$

(D) $B(x)$ must be true

Multiple Choice Question

Let the universe be the natural numbers $(0, 1, 2, \ldots)$. Also, let

$$\begin{aligned} P(x) &= \text{“}x \text{ is prime”} \\ B(x) &= \text{“}x > 2\text{”} \\ O(x) &= \text{“}x \text{ is odd”} \end{aligned}$$

Suppose we want to prove

$$\vdash \quad \forall x[((P(x) \land B(x)) \implies O(x))]$$

Do we have ways to reconstruct facts about equality, multiplication, and addition using only predicate logic?

(A) Yes

(B) No

Multiple Choice Question

Let the universe be the natural numbers $(0, 1, 2, \ldots)$. Also, let

$$P(x) = \text{“}x \text{ is prime”}$$
$$B(x) = \text{“}x > 2\text{”}$$
$$O(x) = \text{“}x \text{ is odd”}$$

Suppose we want to prove

$$\vdash \quad \forall x[((P(x) \land B(x)) \implies O(x))]$$

If we had the right additional axioms & theorems, could we prove this statement?

(A) Yes

(B) No

(C) It depends

Multiple Choice Question

Let the universe be the natural numbers $(0, 1, 2, \ldots)$. Also, let

$$P(x) = \text{“}x \text{ is prime”}$$
$$B(x) = \text{“}x > 2\text{”}$$
$$O(x) = \text{“}x \text{ is odd”}$$

Suppose we want to prove

$$\vdash \quad \forall x[((P(x) \wedge B(x)) \implies O(x))]$$

If we had the right additional axioms & theorems, could we prove any true statement about the natural numbers?

(A) Yes

(B) No

(C) It depends

Multiple Choice Question

Let the universe be the natural numbers $(0, 1, 2, \ldots)$. Also, let

$$P(x) = \text{“}x \text{ is even”}$$
$$B(x) = \text{“}x > 2\text{”}$$
$$O(x) = \text{“}x \text{ is the sum of two primes”}$$

Is the following WFF true?

$$\forall x[((P(x) \wedge B(x)) \implies O(x))]$$

(A) Yes

(B) No

From Formal to Informal

What additional axioms do mathematicians work with?

- Peano Arithmetic
- Euclidean Geometry
- Real Closed Fields (i.e., real numbers)
- ...

Theorem (Gödel's First Incompleteness Theorem)

No consistent axiomatic system (e.g., propositional logic, predicate logic) is capable of proving all truths about the natural numbers.

Reality Check

Formal systems still provide a scaffold on which we base many principles of mathematics, but...

- We're human, and formalisation is pretty technical
- We only have half a quarter left—let's prove interesting things!

Formal Proof vs Informal Proof

Formal Proof

- WFFs & justifications line-by-line
- Systematic
- Relies on appeal to precise axiomatization

Informal Proof

- Natural language (e.g., English)
- Still systematic, but it feels less restrictive
- Relies on underlying, less overt formality

Example: x is even $\qquad \vdash \qquad x+1$ is odd

1. $E(x)$ — Hypothesis
2. $(E(x) \implies \exists k[x = 2k])$ — Definition of "even"
3. $\exists k[x = 2k]$ — Modus Ponens: lines 1 & 2
4. $x = 2\underline{k}$ — Existential Instantiation: line 3
5. $x + 1 = 2\underline{k} + 1$ — Add same quantity to both sides: line 4
6. $\exists k[x + 1 = 2k + 1]$ — Existential Generalization: line 5
7. $(\exists k[x + 1 = 2k + 1] \implies O(x + 1))$ — Definition of "odd"
8. $O(x + 1)$ — Modus Ponens: lines 6 and 7

Ideally, a formal and informal proof of the same thing will mirror each other. Which of the following is the best English analog for line 1?

(A) "x is even" is a true statement.

(B) x is an even number.

(C) It is my opinion that x is even.

(D) Suppose x is an even number.

Example: x is even $\vdash$ $x+1$ is odd

1.	$E(x)$	Hypothesis
2.	$(E(x) \implies \exists k[x = 2k])$	Definition of "even"
3.	$\exists k[x = 2k]$	Modus Ponens: lines 1 & 2
4.	$x = 2\underline{k}$	Existential Instantiation: line 3
5.	$x + 1 = 2\underline{k} + 1$	Add same quantity to both sides: line 4
6.	$\exists k[x + 1 = 2k + 1]$	Existential Generalization: line 5
7.	$(\exists k[x + 1 = 2k + 1] \implies O(x + 1))$	Definition of "odd"
8.	$O(x + 1)$	Modus Ponens: lines 6 and 7

Ideally, a formal and informal proof of the same thing will mirror each other. Which of the following is the best English analog for lines 2–4?

(A) By definition, $x = 2k$.

(B) By definition and Modus Ponens, $x = 2k$.

(C) By definition, $x = 2k$ for some constant k.

(D) By definition and Modus Ponens, $x = 2k$ for some constant k.

Example: x is even $\vdash$ $x+1$ is odd

1. $E(x)$ — Hypothesis
2. $(E(x) \implies \exists k[x = 2k])$ — Definition of "even"
3. $\exists k[x = 2k]$ — Modus Ponens: lines 1 & 2
4. $x = 2\underline{k}$ — Existential Instantiation: line 3
5. $x + 1 = 2\underline{k} + 1$ — Add same quantity to both sides: line 4
6. $\exists k[x + 1 = 2k + 1]$ — Existential Generalization: line 5
7. $(\exists k[x + 1 = 2k + 1] \implies O(x + 1))$ — Definition of "odd"
8. $O(x + 1)$ — Modus Ponens: lines 6 and 7

Ideally, a formal and informal proof of the same thing will mirror each other. Which of the following is the best English analog for lines 5–8?

(A) Adding equal quantities to both sides yields $x + 1 = 2k + 1$.

(B) Adding 1 to both sides yields $x + 1 = 2k + 1$.

(C) Adding 1 to both sides yields $x + 1 = 2k + 1$, so $x + 1$ and $2k + 1$ are odd.

(D) Adding 1 to both sides yields $x + 1 = 2k + 1$, so $x + 1$ is odd by definition.

Guidelines

- The Deduction Theorem
  - Often tacit wherever there's an implication
  - "Assume that [antecedent]. We show that [consequent]."
- Hypotheses & Tautologies
  - "It's clear that..."
  - "We know that..."
  - "Assume that..."
- Modus Ponens
  - Not often associated with excess verbiage
  - "Because [antecedent], it follows that [consequent]."

Guidelines

- Universal Generalization
  - Often implicit: if we discuss a variable, we assume it's arbitrary
  - "Since x was arbitrary, ________ holds true in general."
- Instantiation
  - Existential vs Universal is largely context-dependent
  - "Fix an x such that..."
  - "Let x be..."
  - "Consider an arbitrary x such that ..."
  - "For some constant c, ..."

Multiple Choice Question

Consider once more the Goldbach Conjecture

> "Every even number bigger than 2 is equal to the sum of two primes."

If we wanted to prove this true, what would we need to do?

(A) Let x be an arbitrary even number greater than 2, show it must be equal to two primes

(B) Fix x to a particular even number greater than 2 (e.g., 1024), show it is equal to the sum of two primes

(C) List out as many even xs greater than 2 as possible, confirm that each one is equal to the sum of two primes

(D) It can't be proven

Multiple Choice Question

Consider once more the Goldbach Conjecture

> "Every even number bigger than 2 is equal to the sum of two primes."

If we wanted to prove this false, what would we need to do?

(A) Let x be an arbitrary even number greater than 2, show it can't be equal to two primes

(B) Fix x to a particular even number greater than 2 (e.g., 1024), show it is not equal to the sum of two primes

(C) Let x be an arbitrary number that is odd or ≤ 2, show that it is equal to the sum of two primes

(D) It can't be proven

Multiple Choice Question

Let the universe of discourse be the natural numbers $(0, 1, 2, \ldots)$. Also, let

$$E(x) = \text{“}x \text{ is even”}$$
$$O(x) = \text{“}x \text{ is odd”}$$
$$s(x) = x^2$$

“Behind the scenes”, what is the following sentence saying?

“The square of an even natural number is always odd.”

(A) $\forall x[O(x)]$
(B) $\forall x[O(s(x))]$
(C) $\forall x[(E(x) \wedge O(s(x)))]$
(D) $\forall x[(E(x) \implies O(s(x)))]$

Multiple Choice Question

Let the universe of discourse be the natural numbers $(0, 1, 2, \ldots)$. Also, let

$$E(x) = \text{“}x \text{ is even”}$$
$$O(x) = \text{“}x \text{ is odd”}$$
$$s(x) = x^2$$

“The square of an even natural number is always odd.”

i.e., $\forall x[(E(x) \implies O(s(x)))]$. Suppose we want to prove this sentence is true. What would we need to do?

(A) Let x be an arbitrary even value, show its square must be odd.
(B) Let x be an arbitrary value whose square is odd, show that x is even.
(C) Fix x to a particular even value, show that the square must be odd.
(D) Fix x to a particular value whose square is odd, show that x is even.

Multiple Choice Question

Let the universe of discourse be the natural numbers $(0, 1, 2, \ldots)$. Also, let

$$E(x) = \text{"}x \text{ is even"}$$
$$O(x) = \text{"}x \text{ is odd"}$$
$$s(x) = x^2$$

"The square of an even natural number is always odd."

i.e., $\forall x[(E(x) \implies O(s(x)))]$. Suppose we want to prove this sentence is false. What would we need to do?

(A) Let x be an arbitrary even value, show its square must be even.

(B) Let x be an arbitrary value whose square is even, show that x is even.

(C) Fix x to a particular even value, show that the square must be even.

(D) Fix x to a particular value whose square is even, show that x is even.

Multiple Choice Question

In summary, to prove something of the form

$$\forall x[P(x)]$$

what would be the strategy?

(A) Find a particular x such that $\neg P(x)$

(B) Find a particular x such that $P(x)$

(C) For every x possible, show $P(x)$

(D) For every x possible, show $\neg P(x)$

Multiple Choice Question

In summary, to disprove something of the form

$$\forall x[P(x)]$$

what would be the strategy?

(A) Find a particular x such that $\neg P(x)$

(B) Find a particular x such that $P(x)$

(C) For every x possible, show $P(x)$

(D) For every x possible, show $\neg P(x)$

Multiple Choice Question

Extrapolating, to prove something of the form

$$\exists x[P(x)]$$

what would be the strategy?

(A) Find a particular x such that $\neg P(x)$

(B) Find a particular x such that $P(x)$

(C) For every x possible, show $P(x)$

(D) For every x possible, show $\neg P(x)$

Multiple Choice Question

Extrapolating, to disprove something of the form

$$\exists x[P(x)]$$

what would be the strategy?

(A) Find a particular x such that $\neg P(x)$

(B) Find a particular x such that $P(x)$

(C) For every x possible, show $P(x)$

(D) For every x possible, show $\neg P(x)$

Proof Strategies: Example vs Counterexample

Definition (Proof by Example)

To prove: $\vdash \quad \exists x[P(x)]$

Show: an example—an x such that $P(x)$

Definition (Disproof by Counterexample)

To prove: $\vdash \quad \neg\forall x[P(x)]$

Show: a counterexample—an x such that $\neg P(x)$

Question

Which one of these seems more useful?

(A) Proof by Example

(B) Disproof by Counterexample

Proof Strategies: Exhaustive Proof

Definition (Exhaustive Proof)

To prove: $\vdash \quad \forall x[P(x)]$

Show: $P(\underline{x})$ is true for every possible constant $\underline{x}$

ALSO

To prove: $\vdash \quad \neg\exists x[P(x)]$

Show: $\neg P(\underline{x})$ is true for every possible constant $\underline{x}$

Multiple Choice Question

Which technique would you employ on the following sentence?

Everyone in this classroom is male.

(A) Proof by Example
(B) Disproof by Counterexample
(C) Exhaustive Proof
(D) None of the above

Multiple Choice Question

Which technique would you employ on the following sentence?

Nobody in this classroom is male.

(A) Proof by Example
(B) Disproof by Counterexample
(C) Exhaustive Proof
(D) None of the above

Multiple Choice Question

Which technique would you employ on the following sentence?

Somebody in this classroom is male.

(A) Proof by Example
(B) Disproof by Counterexample
(C) Exhaustive Proof
(D) None of the above

Multiple Choice Question

Which technique would you employ on the following sentence if you assumed it was true?

Everybody in this classroom likes CS 130.

(A) Proof by Example
(B) Disproof by Counterexample
(C) Exhaustive Proof
(D) None of the above

Multiple Choice Question

Which technique would you employ on the following sentence if you assumed it was false?

Everybody in this classroom likes CS 130.

(A) Proof by Example
(B) Disproof by Counterexample
(C) Exhaustive Proof
(D) None of the above

Multiple Choice Question

Which technique would you employ on the following sentence?

No three positive integers a, b, and c can satisfy the equation $a^n + b^n = c^n$ for any integer $n > 2$.

(A) Proof by Example

(B) Disproof by Counterexample

(C) Exhaustive Proof

(D) None of the above

Proof Strategies: Direct Proof

Definition (Direct Proof)

To prove: $\vdash \quad (P \implies Q)$

Show: $P \quad \vdash \quad Q$

Informally,

Proof ("If P, then Q").

Suppose P.

$\vdots$

Therefore Q. □

Multiple Choice Question

In a formal proof, how does proving

$$P \quad \vdash \quad Q$$

prove $\vdash \quad (P \implies Q)$?

(A) The converse of the Deduction Theorem

(B) The Deduction Theorem

(C) The Soundness Theorem

(D) The Completeness Theorem

Multiple Choice Question

Let the universe of discourse be the integers $(\ldots, -2, -1, 0, 1, 2, \ldots)$. Also, let

$$E(x) = \text{"}x \text{ is even"}$$
$$s(x) = x^2$$

How would you symbolize the following sentence?

"The square of an even integer is even."

(A) $\forall x[E(s(x))]$
(B) $\forall x[(E(x) \wedge E(s(x)))]$
(C) $\forall x[(E(x) \implies E(s(x)))]$
(D) $\forall x[(E(s(x)) \implies E(x))]$

Multiple Choice Question

Can we use a direct proof to prove the following sentence?

"The square of an even integer is even."

(A) No, this is a $\forall$ statement
(B) No, direct proof is only available in propositional logic
(C) Yes, just let x be arbitrary (with Universal Instantiation) beforehand
(D) Yes, just fix x to a specific constant beforehand

Multiple Choice Question

Direct Proof ("The square of an even integer is even.")

⋮

□

How should we begin this proof?

(A) Assume x is an even integer

(B) Assume x^2 is an even integer

(C) Let x be an even integer, say 12

(D) Let x^2 be an even integer, say 12

Multiple Choice Question

Direct Proof ("The square of an even integer is even.")

Assume x is an even integer.

$\vdots$

□

Which of the following can we deduce?

(A) By definition, $x = 2k + 1$ for some integer k

(B) By definition, $x = 2k$ for some integer k

(C) By definition, $x = 2^k$ for some integer k

(D) By definition, x^2 is even

Multiple Choice Question

Direct Proof ("The square of an even integer is even.")

Assume x is an even integer. By definition, $x = 2k$ for some integer k.

$\vdots$

$\square$

How can we figure out the value of x^2 from the equation $x = 2k$?

(A) Multiply both sides by x

(B) Multiply both sides by $2k$

(C) Square both sides

(D) Take the square root of both sides

Multiple Choice Question

Direct Proof ("The square of an even integer is even.")

Assume x is an even integer. By definition, $x = 2k$ for some integer k. Squaring both sides yields $x^2 = (2k)^2 = ?$

$\vdots$

□

Which of the following is the most useful equivalent of $(2k)^2$?

(A) x^2

(B) 2^2k^2

(C) $4k^2$

(D) $2(2k^2)$

Multiple Choice Question

Direct Proof ("The square of an even integer is even.")

Assume x is an even integer. By definition, $x = 2k$ for some integer k. Squaring both sides yields $x^2 = (2k)^2 = 2(2k^2)$.

$\vdots$

$\square$

What can we conclude about x^2?

(A) x^2 is even, since it's equal to 4 times an integer (k^2)

(B) x^2 is even, since it's equal to 2 times an integer ($2k^2$)

(C) x^2 is even, since the square of any arbitrary x is equal to $2(2k^2)$

(D) Nothing

Direct Proof Example

Direct Proof ("The square of an even integer is even.")

Assume x is an even integer. By definition, $x = 2k$ for some integer k. Squaring both sides yields $x^2 = (2k)^2 = 2(2k^2)$. By definition, x^2 is even because it's equal to 2 times an integer constant $(2k^2)$. □

Multiple Choice Question

Direct Proof ("Let x be an integer. If x^2 is even, then so is x.")

$\vdots$

$\square$

How should we begin this proof?

(A) Assume x is an integer.

(B) Assume x is an integer such that x^2 is even.

(C) Assume x^2 is an integer.

(D) Assume x^2 is an integer such that x is even.

Multiple Choice Question

Direct Proof ("Let x be an integer. If x^2 is even, then so is x.")

Assume x is an integer such that x^2 is even.

$\vdots$

□

Which of the following can we infer?

(A) By definition, $x^2 = 2k$ for some integer k

(B) By definition, $x = 2k$ for some integer k

(C) By definition, $x^2 = (2k)^2$ for some integer k

(D) None of the above

Multiple Choice Question

Direct Proof ("Let x be an integer. If x^2 is even, then so is x.")

Assume x is an integer such that x^2 is even. By definition, $x^2 = 2k$ for some integer k.

$\vdots$

□

How can we figure out the value of x from the equation $x^2 = 2k$?

(A) Square both sides

(B) Divide both sides by x

(C) Take the square root of both sides

(D) Divide both sides by $2k$

Multiple Choice Question

Direct Proof ("Let x be an integer. If x^2 is even, then so is x.")

Assume x is an integer such that x^2 is even. By definition, $x^2 = 2k$ for some integer k. Taking the square root of both sides yields $x = \sqrt{2k}$.

$\vdots$

□

What can we infer from this?

(A) Something useful

(B) Nothing useful

Proof Strategies: Proof by Contraposition

Definition (Proof by Contraposition)

To prove: $\vdash \quad (P \implies Q)$

Show: $\vdash \quad (\neg Q \implies \neg P)$

$$(\neg B \implies \neg A) \qquad \vdash \qquad (A \implies B) \qquad\qquad \textbf{(Axiom 3i)}$$

In a formal proof, this would be a use of **Axiom 3i** from Homework 2:

1.		
	⋮	
line #.	$(\neg Q \implies \neg P)$	Justification
line #+1.	$(P \implies Q)$	Axiom 3i: line #

Multiple Choice Question

Proof by Contraposition ("Let x be an integer. If x^2 is even, then so is x.")

$\vdots$

□

How should we begin this proof?

(A) Let x be an integer

(B) Let x be an even integer

(C) Let x be an odd integer

(D) Let x be an odd non-integer

Multiple Choice Question

Proof by Contraposition ("Let x be an integer. If x^2 is even, then so is x.")

Let x be an odd integer.

$\vdots$

□

What can we infer from this?

(A) By definition, $x = 2k$ for some integer k

(B) By definition, $x = 2(k+1)$ for some integer k

(C) By definition, $x = 2k+1$ for some integer k

(D) By definition, $x \neq 2$

Multiple Choice Question

Proof by Contraposition ("Let x be an integer. If x^2 is even, then so is x.")

Let x be an odd integer. By definition, $x = 2k + 1$ for some integer k.

$\vdots$

□

How can we figure out the value of x^2 from the equation $x = 2k + 1$?

(A) Square both sides

(B) Multiply both sides by x

(C) Subtract one from both sides, then square both sides

(D) None of the above

Multiple Choice Question

Proof by Contraposition ("Let x be an integer. If x^2 is even, then so is x.")

Let x be an odd integer. By definition, $x = 2k + 1$ for some integer k.
Squaring both sides yields $x^2 = (2k+1)^2 = 4k^2 + 2k + 1 =?$

$\vdots$

□

Which of the following is the most useful equivalent of $4k^2 + 2k + 1$?

(A) $(2k+1)^2$

(B) $(2k)^2 + 2k + 1$

(C) $2(2k^2 + k) + 1$

(D) $4k^2 + k + k + 1$

Multiple Choice Question

Proof by Contraposition ("Let x be an integer. If x^2 is even, then so is x.")

Let x be an odd integer. By definition, $x = 2k + 1$ for some integer k. Squaring both sides yields $x^2 = (2k + 1)^2 = 4k^2 + 2k + 1$
$= 2(2k^2 + k) + 1$.

$\vdots$

□

What can we conclude about x^2?

(A) By definition, x^2 is odd

(B) By definition, x^2 is even

(C) x^2 is either even or odd

(D) Can't say whether x^2 is even or odd

Proof by Contraposition Example

Proof by Contraposition (“Let x be an integer. If x^2 is even, then so is x.”)

Let x be an odd integer. By definition, $x = 2k + 1$ for some integer k. Squaring both sides yields $x^2 = (2k + 1)^2 = 4k^2 + 2k + 1$ $= 2(2k^2 + k) + 1$. By definition, x^2 is odd, since it's equal to $2c + 1$ for the integer constant $c = (2k^2 + k)$. □

Proof Strategies: Proof by Cases

Definition (Proof by Cases)

To prove: $\vdash \quad ((A_1 \vee A_2) \implies B)$

Show: $\vdash \quad (A_1 \implies B)$ **and** $\vdash \quad (A_2 \implies B)$

Informally, where A is equivalent to $(A_1 \vee A_2)$,

Proof by Cases ("If A, then B.")

By cases on A:

$(A = A_1)$ Assume A_1. ... Therefore B.

$(A = A_2)$ Assume A_2. ... Therefore B.

□

Multiple Choice Question

"There exist irrational numbers a & b such that a^b is rational."

What does it mean for a number to be rational?

(A) It is a whole number

(B) It can be expressed as a fraction of two integers

(C) It can be written in decimal notation

(D) When written in decimal notation, it has infinitely many digits

Multiple Choice Question

"There exist irrational numbers a & b such that a^b is rational."

Proof.

Consider $\sqrt{2}^{\sqrt{2}}$. □

Is $\sqrt{2}^{\sqrt{2}}$ rational?

(A) Yes, it is rational

(B) No, it is irrational

(C) I don't really know one way or the other, but one of the above must be true

(D) It's impossible to know one way or the other

Multiple Choice Question

"There exist irrational numbers a & b such that a^b is rational."

Proof.

Consider $\sqrt{2}^{\sqrt{2}}$. We have the following cases:

$\sqrt{2}^{\sqrt{2}}$ is rational: ...

$\sqrt{2}^{\sqrt{2}}$ is irrational: ...

□

Consider the possibility that $\sqrt{2}^{\sqrt{2}}$ is rational.
Is $\sqrt{2}$ rational?

(A) Yes, it is rational
(B) No, it is irrational
(C) I don't really know one way or the other
(D) It's impossible to know one way or the other

Multiple Choice Question

"There exist irrational numbers a & b such that a^b is rational."

Proof.

Consider $\sqrt{2}^{\sqrt{2}}$. We have the following cases:

$\sqrt{2}^{\sqrt{2}}$ is rational: $a = b = \sqrt{2}$, which is irrational

$\sqrt{2}^{\sqrt{2}}$ is irrational: Let $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$. Then, $a^b = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = ?$

□

Consider the possibility that $\sqrt{2}^{\sqrt{2}}$ is irrational.

Which of the following is the most useful property of exponents here?

(A) $(x^y)^z = x^{yz}$

(B) $x^y x^z = x^{y+z}$

(C) Both of the above

(D) None of the above

Multiple Choice Question

"There exist irrational numbers a & b such that a^b is rational."

Proof.

Consider $\sqrt{2}^{\sqrt{2}}$. We have the following cases:

$\sqrt{2}^{\sqrt{2}}$ is rational: $a = b = \sqrt{2}$, which is irrational

$\sqrt{2}^{\sqrt{2}}$ is irrational: Let $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$. Then,

$$a^b = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\times\sqrt{2}} = \sqrt{2}^{?}$$

□

What is $\sqrt{2} \times \sqrt{2}$?

(A) $\sqrt{2}$
(B) 2
(C) $2\sqrt{2}$
(D) 4

Multiple Choice Question

"There exist irrational numbers a & b such that a^b is rational."

Proof.

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$$a^b = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\times\sqrt{2}} = \sqrt{2}^2 = ?$$

□

What is $\sqrt{2}^2$?

(A) $\sqrt{2}$

(B) 2

(C) $2\sqrt{2}$

(D) 4

Multiple Choice Question

"There exist irrational numbers a & b such that a^b is rational."

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$\sqrt{2}^{\sqrt{2}}$ is irrational: Let $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$. Then,

$$a^b = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\times\sqrt{2}} = \sqrt{2}^2 = 2.$$

□

Is 2 rational?

(A) Yes, it is rational

(B) No, it is irrational

(C) I don't really know one way or the other

(D) It's impossible to know one way or the other

Proof Strategies: Proof by Contradiction

Definition (Proof by Contradiction)

To prove: $\vdash \quad P$

Show: $\vdash \quad (\neg P \implies \bot)$—that is, derive a contradiction

Intuitively, if P is false, we could prove (i.e., demonstrate the supposed truth of) something that's false. This is clearly ridiculous, so P must be true.

Informally, to prove P by contradiction,

Proof by Contradiction.

To the contrary, suppose $\neg P$.

$\vdots$

Thus, `[something that's false]`—a contradiction.
Therefore, P must be true. □

Multiple Choice Question

"There is no smallest positive (nonzero) rational number."

Proof by Contradiction.

⋮

□

What do we assume for the sake of contradiction?

(A) There is no smallest positive (nonzero) rational number
(B) There is a smallest positive (nonzero) rational number
(C) There is a smallest positive (possible zero) rational number
(D) There is a smallest negative (nonzero) rational number

Multiple Choice Question

"There is no smallest positive (nonzero) rational number."

Proof by Contradiction.

To the contrary, suppose there is a smallest positive (nonzero) rational number, a/b where a & b are integers > 0.

$\vdots$

□

With of the following is a smaller positive nonzero rational number?

(A) $(a-1)/b$

(B) $a/(b-1)$

(C) $(a+1)/b$

(D) $a/(b+1)$

Proof by Contradiction Example

"There is no smallest positive (nonzero) rational number."

Proof by Contradiction.

To the contrary, suppose there is a smallest positive (nonzero) rational number, a/b where a & b are integers > 0.
$a/(b+1)$ is also a positive, nonzero rational number. However,

$$\frac{a}{(b+1)} < \frac{a}{b}$$

even though a/b was assumed to the the smallest—a contradiction. Thus, there is no smallest positive (nonzero) rational number. □

Multiple Choice Question

Proof by Contradiction ($\sqrt{2}$ is irrational).

$\vdots$

□

What do we assume for the sake of contradiction?

(A) $\sqrt{2}$ is irrational

(B) $\sqrt{2}$ is rational

(C) There is some number x that is irrational

(D) There is some number x that is rational

Multiple Choice Question

Proof by Contradiction ($\sqrt{2}$ is irrational).

To the contrary, suppose $\sqrt{2}$ is rational.

$\vdots$

$\square$

What does $\sqrt{2}$ being rational give us?

(A) $\sqrt{2} = a/b$ for some integers a and b

(B) $\sqrt{2}$ can be written in decimal notation in a finite number of digits

(C) $\sqrt{2}/b$ is equal to some rational number

(D) $\sqrt{2}$ is not the smallest rational number

Multiple Choice Question

Proof by Contradiction ($\sqrt{2}$ is irrational).

To the contrary, suppose $\sqrt{2}$ is rational.
Then, $\sqrt{2} = a/b$ for some integers a and b (where $b \neq 0$).

$\vdots$

□

Would it be possible for a and b to have common factors?

(A) Yes

(B) No

Multiple Choice Question

Proof by Contradiction ($\sqrt{2}$ is irrational).

To the contrary, suppose $\sqrt{2}$ is rational.
Then, $\sqrt{2} = a/b$ for some integers a and b (where $b \neq 0$).

$\vdots$

□

Would it okay to assume a and b have no common factors?

(A) Yes

(B) No

Multiple Choice Question

Proof by Contradiction ($\sqrt{2}$ is irrational).

To the contrary, suppose $\sqrt{2}$ is rational.
Then, $\sqrt{2} = a/b$ for some integers a and b (where $b \neq 0$).
Without loss of generality, a and b have no common factors.

$$\sqrt{2} = \frac{a}{b}$$

□

What should we do now in our search for a contradiction?

(A) Multiply both sides by b

(B) Square both sides

(C) Divide both sides by $\sqrt{2}$

(D) Multiply both sides by $\sqrt{2}$

Multiple Choice Question

Proof by Contradiction ($\sqrt{2}$ is irrational).

To the contrary, suppose $\sqrt{2}$ is rational.
Then, $\sqrt{2} = a/b$ for some integers a and b (where $b \neq 0$).
Without loss of generality, a and b have no common factors.

$$\sqrt{2} = a/b$$
$$2 = a^2/b^2$$
$$2b^2 = a^2$$

□

What does this tell us about a^2?

(A) a^2 is rational

(B) a^2 is irrational

(C) a^2 is even

(D) None of the above

Multiple Choice Question

Proof by Contradiction ($\sqrt{2}$ is irrational).

To the contrary, suppose $\sqrt{2}$ is rational.
Then, $\sqrt{2} = a/b$ for some integers a and b (where $b \neq 0$).
Without loss of generality, a and b have no common factors.

$$\sqrt{2} = a/b$$
$$2 = a^2/b^2$$
$$2b^2 = a^2$$

□

What does a^2 being even tell us about a?

(A) a is even

(B) a is odd

(C) a might be even or odd

(D) None of the above

Multiple Choice Question

Proof by Contradiction ($\sqrt{2}$ is irrational).

To the contrary, suppose $\sqrt{2}$ is rational.
Then, $\sqrt{2} = a/b$ for some integers a and b (where $b \neq 0$).
Without loss of generality, a and b have no common factors.
Facts:

- $2b^2 = a^2$
- a is even—that is, $a = 2k$ for some constant k

□

What should we do with these facts?

(A) Square both sides of the equation $a = 2k$

(B) Substitute $2k$ for a in the equation $2b^2 = a^2$

(C) Substitute $\sqrt{2b^2}$ for a in the equation $a = 2k$

(D) Substitute $2k$ for a in the equation $\sqrt{2} = a/b$

Multiple Choice Question

Proof by Contradiction ($\sqrt{2}$ is irrational).

To the contrary, suppose $\sqrt{2}$ is rational.
Then, $\sqrt{2} = a/b$ for some integers a and b (where $b \neq 0$).
Without loss of generality, a and b have no common factors.
Facts:

- $2b^2 = a^2$
- a is even—that is, $a = 2k$ for some constant k

$$\begin{aligned} 2b^2 &= a^2 \\ 2b^2 &= (2k)^2 \\ 2b^2 &= 2(2k^2) \\ b^2 &= 2k^2 \end{aligned}$$

□

Multiple Choice Question

Proof by Contradiction ($\sqrt{2}$ is irrational).

To the contrary, suppose $\sqrt{2}$ is rational.
Then, $\sqrt{2} = a/b$ for some integers a and b (where $b \neq 0$).
Without loss of generality, a and b have no common factors.
Facts:

- $2b^2 = a^2$
- a is even—that is, $a = 2k$ for some constant k
- $b^2 = 2k^2$

□

What does this tell us about b^2?
(A) b^2 is a rational number involving a
(B) b^2 is even
(C) b^2 is not an integer
(D) b^2 is irrational

Multiple Choice Question

Proof by Contradiction ($\sqrt{2}$ is irrational).

To the contrary, suppose $\sqrt{2} = a/b$ for some integers a and b ($b \neq 0$). WLOG, a and b have no common factors. Then,

- $2b^2 = a^2$
- a is even—that is, $a = 2k$ for some constant k
- $b^2 = 2k^2$
- b is even

□

Where is our contradiction?

(A) a and b have no common factors

(B) a and b are both even

(C) a^2 and b^2 are both even

(D) $\sqrt{2}$ is not actually equal to a/b