Recursion

CS 130

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Winter 2012

Recursion

Definition

Recursion is the process of defining something in terms of itself

- The base case is the simplest instance of the definition, which requires no self-reference
- The recursive case is a more complex instance of the definition, which relies on self-reference to a simpler case (i.e., an instance closer to being the base case)

Example (Recursive Exponentiation)

Suppose we're dealing with natural numbers $(0, 1, 2, \ldots)$.
Exponentiation can be defined recursively upon the operands a and b:

$$a^0 = 1 \qquad \text{(Base Case: } b = 0\text{)}$$

$$a^b = a \times a^{b-1} \qquad \text{(Recursive Case: } b > 0\text{)}$$

Multiple Choice Question

$$a^0 = 1 \qquad \text{(Base Case: } b = 0\text{)}$$
$$a^b = a \times a^{b-1} \qquad \text{(Recursive Case: } b > 0\text{)}$$

Using the recursive definition of exponentiation, what is the value of the following highlighted subexpression?

$$2^3 = ?$$

(A) 8

(B) 2×4

(C) 2×2^2

(D) $2 \times 2^3 - 1$

Multiple Choice Question

$$a^0 = 1 \qquad \text{(Base Case: } b = 0\text{)}$$
$$a^b = a \times a^{b-1} \qquad \text{(Recursive Case: } b > 0\text{)}$$

Using the recursive definition of exponentiation, what is the value of the following highlighted subexpression?

$$2^3 = 2 \times 2^2 = ?$$

(A) 2×2
(B) 2×2^1
(C) 4
(D) $2 \times 2 \times 2^1$

Multiple Choice Question

$$a^0 = 1 \qquad \text{(Base Case: } b = 0\text{)}$$
$$a^b = a \times a^{b-1} \qquad \text{(Recursive Case: } b > 0\text{)}$$

Using the recursive definition of exponentiation, what is the value of the following highlighted subexpression?

$$2^3 = 2 \times 2^2 = 2 \times 2 \times 2^1 = ?$$

(A) 2×2^0
(B) 2×1
(C) 2
(D) $2 \times 2 \times 2 \times 2^0$

Multiple Choice Question

$$a^0 = 1 \qquad \text{(Base Case: } b = 0\text{)}$$
$$a^b = a \times a^{b-1} \qquad \text{(Recursive Case: } b > 0\text{)}$$

Using the recursive definition of exponentiation, what is the value of the following highlighted subexpression?

$$2^3 = 2 \times 2^2 = 2 \times 2 \times 2^1 = 2 \times 2 \times 2 \times 2^0 = ?$$

(A) 2×2^{-1}
(B) 2×1
(C) 2
(D) 1

Multiple Choice Question

$$a^0 = 1 \qquad \text{(Base Case: } b = 0\text{)}$$

$$a^b = a \times a^{b-1} \qquad \text{(Recursive Case: } b > 0\text{)}$$

Using the recursive definition of exponentiation, what is the value of the following highlighted subexpression?

$$2^3 = 2 \times 2^2 = 2 \times 2 \times 2^1 = 2 \times 2 \times 2 \times 2^0 = 2 \times 2 \times 2 \times 1$$

(A) 8

(B) 2^3

(C) Both of the above

(D) None of the above—we don't have a definition for multiplication!

Multiple Choice Question

Let's define multiplication recursively.
Consider multiplying two natural numbers m and n.

$$m \times ? = ? \qquad \text{(Base Case: } n = ?\text{)}$$

$$m \times n = ? \qquad \text{(Recursive Case: ???)}$$

What is the most basic instance of multiplication which requires no recursion?

(A) $n = 1$

(B) $n = 0$

(C) $n = m$

(D) $n = m - 1$

Multiple Choice Question

Let's define multiplication recursively.
Consider multiplying two natural numbers m and n.

$$m \times 0 = 0 \qquad \text{(Base Case: } n = 0\text{)}$$

$$m \times n = ??? \qquad \text{(Recursive Case: } n > 0\text{)}$$

How would we move $n > 0$ closer to the base case?

(A) Add 1 to n

(B) Subtract 1 from n

(C) Divide n by 2

(D) Multiply n by m

Multiple Choice Question

Let's define multiplication recursively.
Consider multiplying two natural numbers m and n.

$$m \times 0 = 0 \qquad \text{(Base Case: } n = 0\text{)}$$
$$m \times n = m \times (n-1) + ??? \qquad \text{(Recursive Case: } n > 0\text{)}$$

So the recursive case must invoke $m \times (n-1)$, as that moves us closer to the base case.
What can we add to the value of $m \times (n-1)$ to get the desired result (i.e., a value equal to the product of m and n)?

(A) 1

(B) n

(C) m

(D) $m \times (n-1)$

Multiple Choice Question

$$m \times 0 = 0 \qquad \text{(Base Case: } n = 0\text{)}$$
$$m \times n = m \times (n - 1) + m \qquad \text{(Recursive Case: } n > 0\text{)}$$

Using the recursive definition of multiplication, what is the value of the following highlighted subexpression?

$$4 \times 3 = ?$$

(A) $4 \times 3 + 3$
(B) $4 \times 2 + 4$
(C) $4 \times 4 + 3$
(D) 12

Multiple Choice Question

$$m \times 0 = 0 \qquad \text{(Base Case: } n = 0\text{)}$$

$$m \times n = m \times (n-1) + m \qquad \text{(Recursive Case: } n > 0\text{)}$$

Using the recursive definition of multiplication, what is the value of the following highlighted subexpression?

$$4 \times 3 = 4 \times 2 + 4 = ?$$

(A) $4 \times 1 + 2$

(B) $4 \times 1 + 4$

(C) $4 \times 1 + 4 + 4$

(D) $4 + 4$

Multiple Choice Question

$$m \times 0 = 0 \qquad \text{(Base Case: } n = 0\text{)}$$
$$m \times n = m \times (n-1) + m \qquad \text{(Recursive Case: } n > 0\text{)}$$

Using the recursive definition of multiplication, what is the value of the following highlighted subexpression?

$$4 \times 3 = 4 \times 2 + 4 = 4 \times 1 + 4 + 4 = ?$$

(A) 4
(B) $4 \times 0 + 1$
(C) $4 \times 0 + 4$
(D) $4 + 0$

Multiple Choice Question

$$m \times 0 = 0 \qquad \text{(Base Case: } n = 0\text{)}$$

$$m \times n = m \times (n - 1) + m \qquad \text{(Recursive Case: } n > 0\text{)}$$

Using the recursive definition of multiplication, what is the value of the following highlighted subexpression?

$$4 \times 3 = 4 \times 2 + 4 = 4 \times 1 + 4 + 4 = 4 \times 0 + 4 + 4 + 4 = ?$$

(A) 4

(B) $4 \times -1 + 4$

(C) 0

(D) 4×0

Multiple Choice Question

$$m \times 0 = 0 \qquad \text{(Base Case: } n = 0\text{)}$$

$$m \times n = m \times (n - 1) + m \qquad \text{(Recursive Case: } n > 0\text{)}$$

Using the recursive definition of multiplication, what is the value of the following highlighted subexpression?

$4 \times 3 = 4 \times 2 + 4 = 4 \times 1 + 4 + 4 = 4 \times 0 + 4 + 4 + 4 = 0 + 4 + 4 + 4 = ?$

(A) 12

(B) 4×3

(C) Both of the above

(D) None of the above—we don't have a definition for addition!

Digression: Turtles All The Way Down

An Apocryphal Tale of Infinite Regress

Old Lady: The world is really a flate plate supported on the back of a giant turtle.

Scientist: And what does the turtle stand on?

Old Lady: Nice try, but it's turtles all the way down!

Multiple Choice Question

Do you suppose we could define addition of natural numbers recursively?

(A) Sure, why not

(B) No, now you're just pushing it

Peano Arithmetic

Definitions (Axioms)

★1. 0 is a natural number
2. For every natural number x, $x = x$
3. For any two natural numbers x and y, if $x = y$ then $y = x$
4. For any three natural numbers x, y, and z, if $x = y$ and $y = z$, then $x = z$
5. For any objects a and b, if a is a natural number and $a = b$, then b is a natural number

★6. For every natural number n, n' is also a natural number
7. For every natural number n, $n' = 0$ is false
8. For all natural numbers m and n, if $m' = n'$ then $m = n$
9. Principle of Induction (we'll cover this later!)

Multiple Choice Question

Let's define addition recursively.

$$m + ? = ? \qquad \text{(Base Case: } n = ?\text{)}$$

$$m + ? = ? \qquad \text{(Recursive Case: } n = ?\text{)}$$

What is the most basic instance of addition which requires no recursion?

(A) $n = 0'$

(B) $n = 1$

(C) $n = 0$

(D) $n = m$

Multiple Choice Question

Let's define addition recursively.

$$m + 0 = ? \qquad \text{(Base Case: } n = 0\text{)}$$
$$m + ? = ? \qquad \text{(Recursive Case: } n = ?\text{)}$$

What should be the value of $m + 0$?

(A) 0

(B) m'

(C) m

(D) n

Multiple Choice Question

Let's define addition recursively.

$$m + 0 = m \qquad \text{(Base Case: } n = 0\text{)}$$

$$m + ? = ? \qquad \text{(Recursive Case: } n = ?\text{)}$$

Intuitively, we want to recurse when $n > 0$.
What does n look like when it's not 0?

(A) $n = 0'$

(B) $n = n'$

(C) $n = m'$

(D) $n = p'$ for some natural number p

Multiple Choice Question

Let's define addition recursively.

$$m + 0 = m \qquad \text{(Base Case: } n = 0\text{)}$$

$$m + p' = ? \qquad \text{(Recursive Case: } n = p'\text{)}$$

When $n = p'$, which of the following values will be closer to the base case of $n = 0$?

(A) n

(B) n'

(C) p

(D) $p' - 1$

Multiple Choice Question

Let's define addition recursively.

$$m + 0 = m \qquad \text{(Base Case: } n = 0\text{)}$$

$$m + p' = \underbrace{m + p}_{?} \qquad \text{(Recursive Case: } n = p'\text{)}$$

Suppose we recursively invoke addition on m and p by saying $m + p$. What do we conceptually need to do to the result of $m + p$ in order to get the proper sum, $m + p'$?

(A) Add 1 to $m + p$

(B) Subtract 1 from $m + p$

(C) Multiply $m + p$ by 2

(D) Divide $m + p$ by 2

Multiple Choice Question

Let's define addition recursively.

$$m + 0 = m \qquad \text{(Base Case: } n = 0\text{)}$$

$$m + p' = \underbrace{m + p}_{?} \qquad \text{(Recursive Case: } n = p'\text{)}$$

Suppose we recursively invoke addition on m and p by saying $m + p$. How would we write the value that is 1 more than $m + p$ using the notation of Peano Arithmetic?

(A) $m + p + 0'$

(B) $(m + p)'$

(C) $m + p'$

(D) $m' + p$

Multiple Choice Question

$$m + 0 = m \qquad \text{(Base Case: } n = 0\text{)}$$
$$m + p' = m' + p \qquad \text{(Recursive Case: } n = p'\text{)}$$

Using the recursive definition of addition, what is the value of the following highlighted subexpression?

$$2 + 3 = ?$$

(A) 5

(B) $0'''''$

(C) $3 + 1$

(D) None of the above

Multiple Choice Question

$$m + 0 = m \qquad \text{(Base Case: } n = 0\text{)}$$
$$m + p' = m' + p \qquad \text{(Recursive Case: } n = p'\text{)}$$

Using the recursive definition of addition, what is the value of the following highlighted subexpression?

$$0'' + 0''' = ?$$

(A) 5
(B) $0'''''$
(C) $0''' + 0''$
(D) $0' + 0''''$

Multiple Choice Question

$$m + 0 = m \qquad \text{(Base Case: } n = 0\text{)}$$
$$m + p' = m' + p \qquad \text{(Recursive Case: } n = p'\text{)}$$

Using the recursive definition of addition, what is the value of the following highlighted subexpression?

$$0'' + 0''' = 0''' + 0'' = ?$$

(A) $0'''' + 0'$
(B) $0'''''$
(C) $0'' + 0'''$
(D) $0' + 0''''$

Multiple Choice Question

$$m + 0 = m \qquad \text{(Base Case: } n = 0\text{)}$$
$$m + p' = m' + p \qquad \text{(Recursive Case: } n = p'\text{)}$$

Using the recursive definition of addition, what is the value of the following highlighted subexpression?

$$0'' + 0''' = 0''' + 0'' = 0'''' + 0' = ?$$

(A) $0''' + 0''$
(B) $0'''''$
(C) $0''''' + 0$
(D) $0' + 0''''$

Multiple Choice Question

$$m + 0 = m \qquad \text{(Base Case: } n = 0\text{)}$$
$$m + p' = m' + p \qquad \text{(Recursive Case: } n = p'\text{)}$$

Using the recursive definition of addition, what is the value of the following highlighted subexpression?

$$0'' + 0''' = 0''' + 0'' = 0'''' + 0' = 0''''' + 0 = ?$$

(A) $0'''''$
(B) 5
(C) $0'' + 0'''$
(D) None of the above

Recursive Algorithms

Definition (Algorithm)

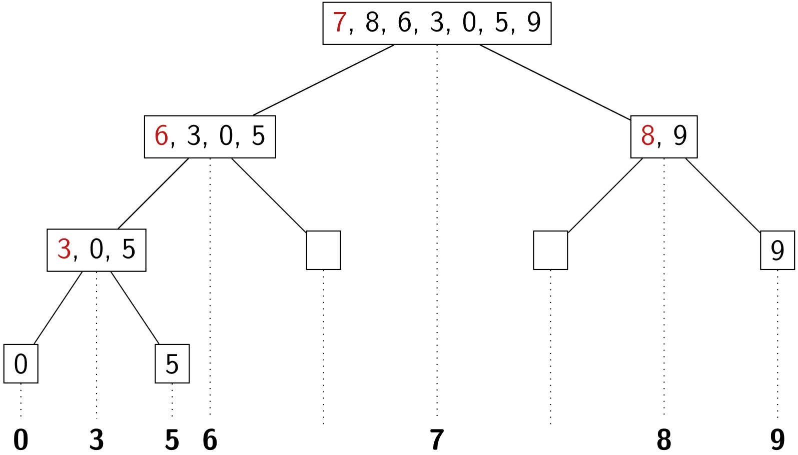
An algorithm is a step-by-step procedure for accomplishing a task

Example (Quicksort)

Consider a sequence of elements that can be compared. To sort the sequence recursively, we can do the following:

- If the sequence has < 2 items, it's sorted (Base Case)
- Otherwise, (Recursive Case)
  - Pick a pivot (typically the leftmost element of the sequence)
  - Recursively sort the left subsequence of items $<$ the pivot
  - Recursively sort the right subsequence of items $>$ the pivot
  - Order thus: left subsequence, pivot, right subsequence

Quicksort Example



Binary Search

Definition

Suppose you want to search through a sorted sequence,

$$S = [S_1, S_2, S_3, \ldots, S_n]$$

for a particular element, x.
In general, you're always searching between two indices: L and R.
Initially, it would be between $L = 1$ and $R = n$.

- Let $M = \lfloor (L + R)/2 \rfloor$
- If $L > R$, then x is not in S (Base Case)
- If $S_M = x$, then x is in S (Base Case)
- If $S_M > x$, then search between L and $M - 1$ (Recursive Case)
- If $S_M < x$, then search between $M + 1$ and R (Recursive Case)

All The Rest

Recursion is used to define a lot of things...

- Certain sequences of numbers (e.g., Fibonacci numbers)
- Language (e.g., the definition of a WFF)
- Collections of things (e.g., natural numbers)
- Data types in some programming languages
- The semantics of computable problems
- ...

So how do we prove facts about recursively-defined things?

Recursive Summation

Definition

Consider a function f upon natural numbers:

$$\sum_{i=s}^{s} f(i) = f(s) \qquad \text{(Base Case: } e = s\text{)}$$

$$\sum_{i=s}^{e} f(i) = f(e) + \sum_{i=s}^{e-1} f(i) \qquad \text{(Recursive Case: } e > s\text{)}$$

Multiple Choice Question

Let's consider how to prove the following equality is true for any natural number, n.

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

Which proof method do you suppose we need to use here?

(A) Proof by Example: when n matches the base case, we can simply plug it in and see if it works

(B) Exhaustive Proof: we need to show it holds true for every possible n

(C) Proof by Cases: depending on whether n matches the base case or the recursive case

(D) Direct Proof: there must be a general way to prove the property

Multiple Choice Question

Let's consider how to prove the following equality is true for any natural number, n.

$$\sum_{i=0}^{n} i \quad = \quad \frac{n(n+1)}{2}$$

At what value of n would $\sum_{i=0}^{n} i$ be equal to its base case?

(A) When $n = i$

(B) When $n = 1$

(C) When $n = 0$

(D) When $e = s$

Multiple Choice Question

Let's consider how to prove the following equality is true for any natural number, n.

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

Consider the case where $n = 0$.

- $\sum_{i=0}^{n} i = ?$
- $n(n+1)/2 = ?$

What is the value of the left side of the equality?

(A) $\sum_{i=0}^{0} i$

(B) $\sum_{i=0}^{n} 0$

(C) $\sum_{i=n}^{0} i$

(D) $\sum_{i=0}^{n} i$

Multiple Choice Question

Let's consider how to prove the following equality is true for any natural number, n.

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

Consider the case where $n = 0$.

- $\sum_{i=0}^{n} i = \sum_{i=0}^{0} i = 0$
- $n(n+1)/2 = ?$

What is the value of the right side of the equality?

(A) $0(0+1)/2$

(B) 0

(C) $\sum_{i=0}^{0} i$

(D) All of the above

Multiple Choice Question

Let's consider how to prove the following equality is true for any natural number, n.

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

Consider the case where $n = 0$.

- $\sum_{i=0}^{n} i = \sum_{i=0}^{0} i = 0$
- $n(n+1)/2 = 0(0+1)/2 = 0$

So, the equality holds when the $\sum$ is at the base case.
What do we know about the value of n when the $\sum$ uses the recursive case?

(A) $n > e$

(B) $n > s$

(C) $n > 0$

(D) $n > i$

Multiple Choice Question

Let's consider how to prove the following equality is true for any natural number, n.

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

Consider the case where $n > 0$.
What is the value of $\sum_{i=0}^{n} i$?

(A) $\sum_{i=0}^{n-1} i$
(B) $i + \sum_{i=0}^{n-1} i$
(C) $0 + \sum_{i=0}^{n-1} i$
(D) $n + \sum_{i=0}^{n-1} i$

Multiple Choice Question

Let's consider how to prove the following equality is true for any natural number, n.

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

$$n + \sum_{i=0}^{n-1} i \stackrel{?}{=} \frac{n(n+1)}{2}$$

Consider the case where $n > 0$.
What is the value of $\sum_{i=0}^{n-1} i$?

(A) $\sum_{i=0}^{n-2} i$

(B) $(n-1) + \sum_{i=0}^{n-2} i$

(C) $0 + \sum_{i=0}^{n-2} i$

(D) None of the above