Relations

CS 130

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Relations

Definition

An n-ary relation on sets $S_1, S_2, \ldots, S_n$ is a subset of $S_1 \times S_2 \times \cdots \times S_n$

n	Relation Type
1	unary
2	binary
3	ternary
4	quaternary

Notation

We'll sometimes say "ρ is a binary relation on S" to mean $\rho \subseteq S \times S$.
We often define a relation with infix notation, like

$$x \; \rho \; y \iff (x, y) \in \rho$$

Multiple Choice Question

Which of the following is a binary relation on $\{1, 2\}$?

(A) $\rho = \{\}$

(B) $\rho = \{(1, 1), (2, 2)\}$

(C) $\rho = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$

(D) All of the above

Multiple Choice Question

Which of the following is a binary relation on $\{1, 2, 3\}$?

(A) $\rho = \{\}$

(B) $\rho = \{(1, 1, 1)\}$

(C) $\rho = \{(1, 1, 1), (1, 1, 2), (1, 1, 3)\}$

(D) All of the above

Multiple Choice Question

Which of the following is a ternary relation on $\{1, 2, 3\}$?

(A) $\rho = \{\}$

(B) $\rho = \{(1, 1, 1)\}$

(C) $\rho = \{(1, 1, 1), (1, 1, 2), (1, 1, 3)\}$

(D) All of the above

Multiple Choice Question

Which of the following defines a binary relation on $\mathbb{N}$?

(A) $x \, \rho \, y \iff (x, y) \in \rho$

(B) $x \, \rho \, y \iff x + y$

(C) $x \, \rho \, y \iff y = x^2$

(D) $x \, \rho \, y \iff \rho \subseteq \mathbb{N} \times \mathbb{N}$

Multiple Choice Question

Which of the following sets defines the relation $\leq$ on $\mathbb{N}$?

(A) $x \; \rho \; y \iff x \leq y$

(B) $\rho = \{(x, y) \mid x \leq y\}$

(C) $\rho = \{(0,0), (0,1), (0,2), \ldots, (1,1), (1,2), (1,3) \ldots\}$

(D) Any of the above

{One,Many}-to-{One,Many}

Note

To save room, we write

$$\forall x, y \in S[\ldots]$$

instead of

$$\forall x \in S[\forall y \in S[\ldots]]$$

Definitions

Let ρ be a binary relation on S

`One-to-One:` $\forall x_1, x_2, y \in S[((x_1 \ \rho \ y) \wedge (x_2 \ \rho \ y)) \implies x_1 = x_2]$

`One-to-Many:` $\exists x, y_1, y_2 \in S[(x \ \rho \ y_1) \wedge (x \ \rho \ y_2) \wedge (y_1 \neq y_2)]$

`Many-to-One:` $\exists x_1, x_2, y \in S[(x_1 \ \rho \ y) \wedge (x_2 \ \rho \ y) \wedge (x_1 \neq x_2)]$

`Many-to-Many:` Both one-to-many and many-to-one

Multiple Choice Question

Which choice describes the following relation on $S = \{2, 5, 7, 9\}$?

$$\rho = \{(5, 2), (7, 5), (9, 2)\}$$

(A) One-to-one

(B) One-to-many

(C) Many-to-one

(D) Many-to-many

Multiple Choice Question

Which choice describes the following relation on $S = \{2, 5, 7, 9\}$?

$$\rho = \{(2, 5), (5, 7), (7, 2)\}$$

(A) One-to-one

(B) One-to-many

(C) Many-to-one

(D) Many-to-many

Multiple Choice Question

Which choice describes the following relation on $S = \{2, 5, 7, 9\}$?

$$\rho = \{(7, 9), (2, 5), (9, 9), (2, 7)\}$$

(A) One-to-one
(B) One-to-many
(C) Many-to-one
(D) Many-to-many

Multiple Choice Question

Which choice describes the following relation on $S = \{2, 5, 7, 9\}$?

$$\rho = \{\}$$

(A) One-to-one
(B) One-to-many
(C) Many-to-one
(D) Many-to-many

Multiple Choice Question

Which choice describes the following relation on the set of humans (living or dead)?

$$\rho = \{(h_1, h_2) \mid h_1 \text{ is the parent of } h_2\}$$

(A) One-to-one

(B) One-to-many

(C) Many-to-one

(D) Many-to-many

Multiple Choice Question

Let $\rho \subseteq \mathbb{N} \times \mathbb{N}$ be defined by

$$\rho = \{(x, y) \mid y = \sqrt{x}\}$$

Is ρ one-to-one?

(A) Yes
(B) No
(C) Maybe
(D) I don't know

Multiple Choice Question

Let $\rho \subseteq \mathbb{N} \times \mathbb{N}$ be defined by

$$\rho = \{(x, y) \mid y = \sqrt{x}\}$$

Does ρ map every “input” (x) to an “output” (y)?

(A) Yes

(B) No

(C) Maybe

(D) I don’t know

Functions

Let ρ be a binary relation on set S.

Definition (Onto)

ρ is onto if $\forall y \in S[\exists x \in S[(x, y) \in \rho]]$

Definition (Inverse)

The inverse (or reversal) of ρ, denoted ρ^{-1}, is the relation

$$\rho^{-1} = \{(y, x) \mid (x, y) \in \rho\}$$

Functions

Definition (Function)

A relation $f \subseteq S \times T$ is a function if f^{-1} is one-to-one and onto

- $f \subseteq S \times T$ can be denoted $f\colon S \to T$
- $(x, y) \in f$ can be denoted $f(x) = y$

A 1-1 & onto function is sometimes called a *1-1 mapping*, a *1-1 correspondence*, or a *bijection*

Definitions

For a function $f\colon S \to T$

- S is the domain of f, denoted $\text{dom}(f)$
- T is the codomain of f, denoted $\text{codom}(f)$
- The range of f is the set $\text{range}(f) = \{f(x) \mid x \in \text{dom}(f)\}$

Multiple Choice Question

Suppose there is a bijection between two sets, $f : X \to Y$. Is it possible $|X| \neq |Y|$?

(A) Yes (B) No (C) I don't know

Comparing Cardinalities

Definition

Two sets X and Y are called equipollent if they have the same cardinalities
That is, X and Y are equipollent if there exists a bijection $f : X \to Y$

Example

Let

$$X = \{1, 3, 5, 7, 9\}$$
$$Y = \{2, 4, 6, 8, 10\}$$

Clearly, they have the same cardinalities (5), but equivalently, we can define the bijection $f : X \to Y$

$$f(x) = x + 1$$

Multiple Choice Question

Consider the sets $\mathbb{N}$ and $\mathbb{Z}$:

$$\begin{array}{ccccccccccccc} & & & & & \{0 & 1 & 2 & 3 & 4 & 5 & \ldots\} & (\mathbb{N}) \\ \{\ldots & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & \ldots\} & (\mathbb{Z}) \end{array}$$

Do you suppose they have the same cardinality?

(A) No, $|\mathbb{Z}| > |\mathbb{N}|$

(B) No, $\mathbb{N} \subseteq \mathbb{Z}$

(C) Yes, they're both infinite

(D) None of the above

Multiple Choice Question

Consider the sets $\mathbb{N}$ and $\mathbb{Z}$:

$$\begin{array}{llllllllll} \{0 & 1 & 2 & 3 & 4 & 5 & 6 & \ldots\} & & (\mathbb{N}) \\ \{0 & -1 & 1 & -2 & 2 & -3 & 3 & \ldots\} & & (\mathbb{Z}) \end{array}$$

Do you suppose they have the same cardinality?

(A) No, $|\mathbb{Z}| > |\mathbb{N}|$

(B) No, $\mathbb{N} \subseteq \mathbb{Z}$

(C) Yes, they're both infinite

(D) Yes, we can find a bijection $f(n) = \begin{cases} \frac{n}{2} & n \text{ is even} \\ -\lceil \frac{n}{2} \rceil & n \text{ is odd} \end{cases}$

Multiple Choice Question

Suppose we have two infinite sets X and Y, but that it was impossible to define a bijection $f\colon X \to Y$.

Could we say X and Y have the same cardinality?

(A) No, there must be a bijection

(B) Yes, they're both infinite

(C) We might be able to

(D) What is "cardinality", anyway, really?

Countability

Definitions

- S is denumerable if it is equipollent to $\mathbb{N}$
- S is countable if it is finite or denumerable
- S is uncountable if it is not countable

Yay!

It's possible to prove $\{x \mid x \in \mathbb{R} \wedge 0 \leq x < 1\}$ is uncountable!

Boo!

We probably won't have time to cover this in class.

Properties of Binary Relations

Let ρ be a binary relation on a set S

Definitions

ρ is...

- **reflexive** if $\forall x \in S[x \; \rho \; x]$
- **symmetric** if $\forall x, y \in S[x \; \rho \; y \implies y \; \rho \; x]$
- **transitive** if $\forall x, y, z \in S[((x \; \rho \; y) \wedge (y \; \rho \; z)) \implies x \; \rho \; z]$
- **antisymmetric** if $\forall x, y \in S[((x \; \rho \; y) \wedge (y \; \rho \; x)) \implies x = y]$
  I.e., $\forall x, y \in S[x \neq y \implies ((x, y) \notin \rho \vee (y, x) \notin \rho)]$

Orderings

Definition (Equivalence Relation)

A relation is an equivalence relation if it is reflexive, symmetric, and transitive

Definition (Partial Ordering)

A relation is a partial ordering if it is reflexive, transitive, and antisymmetric

A poset (partially-ordered set) is a set and a partial order on the set, $(S, \sqsubseteq)$

Definition (Total Ordering)

A relation is a total ordering if it is a partial ordering and every element is comparable—i.e.,

$$\forall x, y \in S[(x, y) \in \rho \vee (y, x) \in \rho]$$

Multiple Choice Question

Consider the relation on $S = \{1, 2, 3\}$ defined by

$$\rho = \{(1,3),(3,3),(3,1),(2,2),(2,3),(1,1),(1,2)\}$$

Reflexive	Symmetric	Antisymm.	Transitive	Equiv.	P.O.	T.O.

(A) Yes

(B) No

(C) I don't know

Multiple Choice Question

Consider the relation on $S = \{1, 2, 3\}$ defined by

$$\rho = \{(1,1), (3,3), (2,2)\}$$

Reflexive	Symmetric	Antisymm.	Transitive	Equiv.	P.O.	T.O.

(A) Yes

(B) No

(C) I don't know

Multiple Choice Question

Consider the relation on $S = \{1, 2, 3\}$ defined by

$$\rho = \{(1,1),(1,2),(1,3),(2,3),(3,1),(1,3)\}$$

Reflexive	Symmetric	Antisymm.	Transitive	Equiv.	P.O.	T.O.

(A) Yes

(B) No

(C) I don't know

Multiple Choice Question

Consider the relation on $S = \{1, 2, 3\}$ defined by

$$\rho = \{(1,1), (1,2), (2,3), (1,3)\}$$

Reflexive	Symmetric	Antisymm.	Transitive	Equiv.	P.O.	T.O.

(A) Yes

(B) No

(C) I don't know

Multiple Choice Question

Consider the set of Booleans, $\mathbb{B} = \{\texttt{T}, \texttt{F}\}$.
What Boolean operator defines the following relation?

$$\rho = \{(\texttt{F}, \texttt{F}), (\texttt{T}, \texttt{F}), (\texttt{T}, \texttt{T})\}$$

(A) $\rho = \{(a, b) \mid a \wedge b\}$
(B) $\rho = \{(a, b) \mid a \vee b\}$
(C) $\rho = \{(a, b) \mid a \implies b\}$
(D) $\rho = \{(a, b) \mid a \iff b\}$

Multiple Choice Question

Consider the $\implies$ relation on $\mathbb{B} = \{\mathtt{T}, \mathtt{F}\}$.

Reflexive	Symmetric	Antisymm.	Transitive	Equiv.	P.O.	T.O.

(A) Yes

(B) No

(C) I don't know

Lattices

Given a poset $(S, \sqsubseteq)$ and a subset $A \subseteq S$ ...

Definition (Lower/Upper Bounds)

$x \in S$ is...

- an upper bound of A (denoted $A \sqsubseteq x$) if $\forall a \in A[a \sqsubseteq x]$
- a lower bound of A (denoted $x \sqsubseteq A$) if $\forall a \in A[x \sqsubseteq a]$

Definition (Least Upper Bound / Greatest Lower Bound)

$x \in S$ is...

- the LUB of A ($x = \bigvee A$) if $A \sqsubseteq x$ and $\forall y \in S[A \sqsubseteq y \implies x \sqsubseteq y]$
- the GLB of A ($x = \bigwedge A$) if $x \sqsubseteq A$ and $\forall y \in S[y \sqsubseteq A \implies y \sqsubseteq x]$

Definition (Lattice)

A lattice is a poset such that every subset has a unique LUB and a unique GLB