CS 130 Midterm Exam

Alex Vondrak

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1 Consider an arbitrary implication $(P \implies Q)$. We said earlier in the quarter that the negation $\neg(P \implies Q)$ is distinct from the contrapositive $(\neg Q \implies \neg P)$.

(a) (5 points) Prove via truth table that $((\neg Q \implies \neg P) \iff \neg(P \implies Q))$ is a contradiction.

(b) (5 points) Explain why $\neg(P \implies Q)$ is actually same thing as $(P \wedge \neg Q)$ by known propositional logic equivalences.

2 Most presentations of propositional logic like to elide parentheses. Let's investigate why this is a reasonable thing to do for $\wedge$: we'll see that $(A \wedge (B \wedge C))$ is the same thing as $((A \wedge B) \wedge C)$, so it doesn't matter that we put any parentheses at all—we can just say $(A \wedge B \wedge C)$.

(a) (2 points) Write the unabbreviated forms of $(A \wedge (B \wedge C))$ and $((A \wedge B) \wedge C)$. Do you think it would be easy to prove (b) & (c) using the unabbreviated forms, or would you rather use results from the list of theorems provided in class?

(b) (4 points) Using propositional logic, prove $(A \wedge (B \wedge C)) \quad \vdash \quad ((A \wedge B) \wedge C)$.

(c) (4 points) Using propositional logic, prove $((A \wedge B) \wedge C) \quad \vdash \quad (A \wedge (B \wedge C))$.

3 Consider the following.

$$(A \implies (B \implies (C \implies D))) \quad \vdash \quad (A \implies ((B \implies C) \implies (B \implies D)))$$

(a) (5 points) Prove this using propositional logic **with** the Deduction Theorem.

(b) (5 points) Prove this using propositional logic **without** the Deduction Theorem.

4 (a) (5 points) Using predicate logic, prove

$$\vdash \quad (\forall x[A(x,x)] \implies \forall x[\exists y[A(x,y)]])$$

(b) (5 points) Prove the converse of the above implication is invalid by giving a model in which it's false.

See other side

Formalism is the philosophy that asserts math is a "meaningless game" involving "meaningless symbols". Really, formal rules don't have to tell us *anything*; we'll attribute meaning on our own. To take this example to the extreme, let's study the **Lucky Charms Axiom System**.

The "meaningless symbols" we'll be playing the "game" with are four of the Lucky Charms marshmallow shapes:

$$\heartsuit \qquad \star \qquad \mho \qquad ☾$$

(heart, star, horseshoe, and moon). For simplicity, we'll ignore clovers, pots of gold, rainbows, balloons, and promotional marshmallow shapes. Plus, it's harder to write those symbols, so these four will suffice.

The goal of the game is, of course, to produce theorems. A theorem is any of the following:

Axiom 1: $\forall x[(\heartsuit x \implies ☾x)]$

Axiom 2: $(\exists x[☾x] \implies \heartsuit\mho)$

Axiom 3: $☾\star$

or any other formula obtainable using the above and a finite sequence of inference rules:

Modus Ponens: If A is a theorem and $(A \implies B)$ is a theorem, then B is also a theorem.

Universal Instantiation: If A is a theorem involving $\forall x$, then the formula obtained by removing the surrounding $\forall x[\ldots]$ and replacing every x with a particular Lucky Charms shape is also a theorem.

Existential Generalization: If A is a theorem involving a particular Lucky Charms shape, then the formula obtained replacing all instances of the marshmallow shape with x and wrapping the formula in $\exists x[\ldots]$ is also a theorem.

5 (10 points) Prove the following theorem using the Lucky Charms Axiom System.

Therorem. $☾\mho$.